Dietmar Herrmann

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History of Mathematics in Ancient Greece and Hellenism

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Anzing, Germany

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Preface

From whoever wants to deliver the history of any knowledge, we can rightly demand that he give us news of how the phenomena became known in due course, what has been fantasized about them, what has been meant and thought.[1]

Even though the *history of mathematics* is not a relevant lecture course at German universities, a supplementary book can be of interest. It offers for all mathematics teachers and -interested people quite a new view on the diverse problems, which were developed over the course of one millennium (Thales 580 BC until Proklos 420 AD) in ancient Greek mathematics. For reasons of space, only facets of the different works can be presented, which however assemble to a kaleidoscope of science. A broad spectrum of tasks, constructions and historical illustrations are put together to a new overall picture, which provides more insight than conventional summary descriptions.

The versatile methods, which the Greek researchers have invented, also demand respect and recognition from today's viewer. These amazing achievements have been created without any aids like calculators and modern communication. It was important to describe the whole range of Greek mathematics, especially by means of literary sources and a well-chosen selection of pictures also to bring in the context of Pythagorean-Platonic philosophy. There are three possibilities of a historical processing: strictly chronological, biographical-person related or subject-related by means of special thematic areas. The present representation chooses a mixture of the latter two.

A first problem in the representation of ancient mathematics is raised by the famous article *On the Need to Rewrite the History of Greek Mathematics* by *Sabetai Unguru*. The author expresses the opinion that it is basically inappropriate to present ancient findings with modern formulas. The formula and concept apparatus of modern mathematics included concepts and abstractions, which might veil the authentic historical procedure. As an example the binomial formula $(a+b)^2 = a^2 + 2ab + b^2$ is chosen. In modern mathematics, it applies to all abstract elements of a commutative ring; such a concept is completely foreign

[1] *J.W. von Goethe: From the preface of the Color Theory (1810).*

to Euclid. A product of two numbers or a square is always associated with an area in Euclid and can only be linked with quantities of the same dimension. The Greek word αριθμος *(arithmos)* must be seen in the Pythagorean-Platonic environment and cannot be adequately translated with the word *number*. In order to make the representations readable and compact, the usual formula language is used.

A second goal is the description of the political and cultural environment in which the Greek scientist is located. The cultural flourishing of Athens in a phase of relative peace between the Persian Wars—due to its leading role in the alliance against the Persians—made it possible to build an academy that attracted educationally motivated people from all over Greece, like Aristotle. Alexander freed Egypt from Persian occupation and caused a shift of power to the southeast. The Egyptian-Syrian province that emerged after his death through the division of the empire became the intellectual and economic center of the Mediterranean region with its capital Alexandria. The schools founded there, the Museion and the Serapeion, survived the collapse of the Ptolemaic Empire and flourished even under Roman occupation. Only the rise of Christianity as the state religion ended the fate of the scientists still clinging to Platonic doctrine, as seen in the fate of Hypatia.

Another concern is the inclusion of new, critical perspectives in comparison to older literature. Stories that the vegetarian Pythagoras sacrificed several bulls when discovering a theorem or that Archimedes set fire to the sails of the Roman fleet with mirrors can be dismissed as fairy tales. A modern interpretation of Diophantus, criticism of the work of Claudius Ptolemy and Heron, as well as new translations of Nicomachus and Theon of Smyrna provide a new view of Greek mathematics. The extensive work of Pappus is completely re-evaluated. The methods used usually require only moderate knowledge.

Geometry is currently taking a back seat in education; but this is not a sufficient reason to completely abolish Euclidean geometry according to the motto of J. Dieudonné (member of the Bourbaki group) *Euclid must go!*

To the 2nd edition The author is indebted to the publisher for the publication of the book now in its 2nd, revised edition. In this way, a new, unique chapter on Roman mathematics could be introduced and illustrated with new images. Many other chapters have been updated and extended, in particular the section dedicated to the continued influence of Hellenistic mathematics in Byzantium and Baghdad.

Furthermore, I would like to thank Professor Lothar Profke for his helpful comments on the first edition. Special thanks go to the program planner Dr. Annika Denkert for her support of the project!

In a variation of the poet Horace's quote *aut prodesse volunt aut delectare poetae*, the author wishes "benefit and/or pleasure" in reading!

Dietmar Herrmann

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Introduction

1

1.1 To the Content of the Book

Chapter 2 describes the emergence of a new Greek culture, which leads to a new beginning of Greek civilization, which later becomes predominant in all of Europe. With the emergence of the sciences, mathematics also develops.

The Chaps. 3 to 5 deal with the beginnings of mathematics by the pioneers Thales, Pythagoras and Hippocrates, all of whom come from the Ionian Islands or coastal cities (Asia minor).

The Chaps. 6 to 8 report how Athens becomes a scientific center through the establishment of the Academy and the Lykeion. Although Plato and Aristotle were not actual mathematicians, they had a decisive impact on mathematics.

After the death of Alexander the Great, his realm of rule disintegrates into individual satrapies and diadoch kingdoms. As the symbiosis of Greek and Egyptian culture under the Ptolemaic royal house makes the new capital of Alexandria a center of trade and science, Chap. 9 describes. Alexandria offers the ideal place of activity for a whole range of famous mathematicians up to the Roman conquest, such as Euclid (Chap. 10), Archimedes (Chap. 12), Eratosthenes (Chap. 13) and Apollonius (Chap. 15).

The three classical problems of cube doubling, angle trisection and squaring the circle are summarized as topics in Chap. 11. Attached are the constructibility of the regular polygons and the quadrature of the so-called lunes, which Hippocrates cleverly developed to find the quadrature of the circle.

Chapter 14 provides an overview of the geometry of conic sections, which can no longer be found in the curricula of secondary schools. It serves as a preparation for the following chapter on Apollonius of Perga (Chap. 15).

Even after its incorporation into the Roman Empire, Alexandria continued to function as a training center and workshop for famous natural scientists. Astronomers such as Aristarchus and Hipparchus (Chap. 16) and Claudius Ptolemy (Chap. 18), engineers like Ctesibius and Heron (Chap. 17) and the

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mathematicians Menelaus, Diophantus (Chap. 21), Pappus of Alexandria (Chap. 22) can be mentioned here. Also discussed are the less well-known scholars such as Nicomachus of Gerasa (Chap. 19), Theon of Smyrna (Chap. 20) and Theon of Alexandria (Chap. 23), who were not primary mathematicians.

As already mentioned in the first part of the preface, it is problematic to describe ancient mathematics with modern formulas; the reader is reminded of the fact by the indication *in modern notation*. For reasons of readability and conciseness of the text, it is not possible to follow all the steps *in the old style*; therefore, differential calculus is used in Chap. 12 to shorten Archimedes' extensive proofs that require numerous other auxiliary theorems. Chap. 11 avoids the application of Galois theory when discussing the classical problems. The use of the trigonometric functions unknown to the Greeks could not be completely avoided, as the exclusive use of the chord function makes reading the text more difficult.

In references to Euclid, Apollonius, etc., the Roman numerals always give the book, the Latin the theorem or paragraph. Euclid (I, 47) is therefore Theorem 47 in the first book of *Elements,* the well-known Pythagorean theorem. Comments and explanations by the author are in square brackets. The references to Plato and Aristotle are given in the usual numbering according to Stephanus or Bekker.

This book is made up of notes and observations that the author has collected over several years in the hope of presenting the material in a well-readable, historically illustrated volume in a modern, critical representation. Of course, it is impossible to list all the mathematical achievements of this millennium; for reasons of space, a representative selection is made of certain, typical questions for each author. A wealth of constructions, problems and algorithms is presented, which should encourage self-study and use in teaching. A large number of illustrations facilitate understanding of the material. To what extent it has been possible to put together the mosaic of ancient Greek mathematics piece by piece and to make their scholars come alive in their socio-cultural, political and religious context is for the reader to decide.

Supplement to the 2nd edition

The author thanks the publisher for publishing the book in a 2nd edition; this allows the author to update the text, include a variety of new color images, and correct a variety of printing errors. New is the Chap. 25 “Roman Mathematics”, a topic that is presented here comprehensively and in a new way for the first time. Also new is the Sect. 16.4 “Applications in Geography”. The presentation of Greek mathematics was significantly extended in the Sects. 2.2 to 2.3, also “The Mathematics of the Early Middle Ages” in Sect. 26.2. For reasons of space, the previous “What Euclid didn’t know yet” had to be omitted. In particular, the text was updated in Chap. 4 (Pythagoras), Chap. 8 (Aristotle and the Lyceum), Chap. 9 (Alexandria), Chap. 12 (Archimedes) und Chap. 21 (Diophantus).

On the English Edition The publisher agreed to have the book automatically translated as part of an AI project. For printing reasons, only a few changes could be made. Therefore, some quotes are re-translated from German. Special

characters – like Greek accents – could not be edited. For interested readers reference is made to further books of the author:

- *Mathematik im Mittelalter:* History of Mathematics in the Occident (with its Sources in China, India and Islamic Culture, Springer).
- *Mathematik im Vorderen Orient:* History of Mathematics in the Near East (Ancient Egypt and Mesopotamia, Springer). Another Book "*Mathematik der Neuzeit*" : History of Mathematics in modern times (from Vieta to Euler) is in print.

1.2 On the State of Mathematical Historical Research

> *I believe that the attempt to teach mathematics without reference to its cultural, social, philosophical and historical background is a serious mistake and a strategic error.*[1]

About the so-called *geometric algebra* wrote O. Neugebauer[2]:

> The answer to the question of where the origin of all fundamental problems in geometric algebra lies [namely in the plane transformations of Euclid (II, 1–10) or Euclid (VI, 24–29)], can be given completely today: on the one hand, they lie in the needs of the Greeks to secure the general validity of their mathematics in the wake of the emerging irrationalities, on the other hand in the resulting necessity to translate the results of pre-Greek algebra. Once the problem is formulated in this way, everything turns out to be trivial and provides a seamless transition from Babylonian algebra to Euclid's formulations.

This view that Euclidean transformations of surfaces represent a form of hidden algebra was widely accepted, as can be seen from the writings of H.G. Zeuthen and B.L. van der Waerden. Zeuthen writes similarly in his treatise[3] about conic sections:

> Although the Greeks did not have the concept of a coordinate system, they would use rectangular and oblique coordinates … The theory of proportions would allow them to carry out the most important algebraic operations. […] The geometric algebra had reached such a point in Euclid's time that it could perform the same tasks as our algebra, as long as this did not go beyond the treatment of expressions of second degree(!).

B.L. van der Waerden also sets the aforementioned surface transformation of Euclid equal to the application of today's binomial formulas like $(a+b)^2 = a^2 + b^2 + 2ab$ in his well-known book *Science Awakening* (p. 119).

[1] R.L. Hayes, 6th International Congress on Mathematical Education, Budapest 1988.

[2] O. Neugebauer: *Zur geometrischen Algebra,* Quellen u. Studien zur Geschichte d. Mathematik, Abt. B3, 254–259.

[3] H.G. Zeuthen: *The Theory of Conic Sections in Antiquity,* Copenhagen 1886.

The Israeli Sabetai *Unguru* put an end to this view with a fundamental article[4]. A large part of his attack concerned the long-established doctrine of Greek *geometric algebra.* A predecessor in this debate was Unguru in Jacob *Klein* (p. 5), who already wrote in 1965:

> Most historical accounts try to grasp Greek mathematics using modern symbolism, as if this were just an external form that could be tailor-made for any content. Even if the research is based on a true understanding of Greek science, it will be seen that the investigation is taking place at a level of knowledge that is shaped by modern ideas.

A. Szabó, who in his book (p. 457) already turned against the theses of O. Neugebauer in 1969:

> (1) Even if we believe that there really was a Babylonian algebra, up to now no one has been able to make it probable with any specific task that the Greeks knew such an algebra in pre-Euclidean times.
> (2) Those propositions in Euclid that we are accustomed to regard as algebraic propositions in geometric guise have, in reality, only so much to do with algebra that we can readily give our algebraic equivalents for these propositions.

Unguru called the term geometric algebra a *fantasy creature, a monstrous hybrid, that mathematicians have invented, who lack any sense of history.* This term must under no circumstances be applied to Babylonian or Greek mathematics.

> This historiographic approach, which is hidden behind the term "geometric algebra", is offensive, naive and historically untenable. To view historical mathematics texts under the perspective of modern mathematics is the surest way to misunderstand the nature of ancient mathematics, in which philosophical preconceptions and metaphysical entanglements have played a much more fundamental and significant role than in modern mathematics. The assumption that one could automatically and indiscriminately apply modern algebraic symbolism to any mathematical content is the surest way to misunderstand the inherent differences that are included in the mathematics of past centuries. Geometry is not algebra!

Later, he added at the same place

> It is regrettable and sad when a student of ancient cultural history has to get to know the terminology and operations of modern mathematics first in order to understand what meaning and intentions modern commentators interpret into the old texts. … The goal of these so-called historical studies is probably to show how the ancient mathematicians hid their modern ideas and procedures under a cloak of awkward, embarrassing, antiquated and old-fashioned expressions. In other words, it is probably the task of a mathematics historian to untangle the old mathematical texts, to translate them into the modern language of mathematics, so that they are available to all interested parties.

[4] S. Unguru: *On the Need to Rewrite the History of Greek Mathematics:* Archive for the History of Exact Sciences 15 (1975), 67–114.

His statement that this approach is *anachronistic* and *unhistorical* and therefore the whole Greek mathematics has to be rewritten, caused furious reactions. Hans *Freudenthal,* Andre *Weil* and B.L. *van der Waerden* published their answers in the same journal; the protest led to the fact that the editorial staff of the journal excluded further contributions from Unguru.

In his counteroffensive, van der Waerden did not hold back:

> Unguru, like many non-mathematicians, greatly overestimates the importance of symbolism in mathematics. These people see our contributions full of formulas and think that these formulas make up the essential content of mathematical thinking. We, the active mathematicians, know better that in many cases the formulas do not represent the essential content, but are only convenient aids.

In a letter to the editor of the journal, A. Weil formulated[5]:

> It is advisable to master mathematics before dealing with its history. […] Books VII to IX of Euclid do not contain any algebra or so-called geometric algebra. Of course, it is much more convenient to operate the algebraic operations with our algebra symbols than with words, as Euclid does; just as it is easier to calculate with decimal fractions (or with computers in binary) than with Archimedes' fractions, that does not change the essence of the matter.

Weil's letter ends with the following words, which one seldom finds in a mathematical journal:

> If a scientific discipline, which intervenes between two already existing (let them be called A and B), establishes itself in a certain sense, this often creates space for the emergence of parasites, who are ignorant in A and B, but try to live off of them by intimidating those working in A, that they would understand nothing of B and vice versa. Unfortunately, we see that this is exactly what is happening at the present time in the history of mathematics. Let us try to stop this infection before it becomes our fate.

Unguru answers four years later in the journal Isis[6]:

> The history of mathematics has usually been written in order to illustrate the proverb "Anachronism is no sin." Most contemporary historians of mathematics, mathematicians since their student days, take it for granted silently or explicitly that mathematical entities come from the world of Platonic ideas, where they patiently wait to be discovered by the genius of an active mathematician. Mathematical concepts, both constructive and computational, are seen as eternal and unchangeable, unaffected by the inherent features of the culture in which they occur; each one is clearly identifiable in its various historical appearances, since all these appearances are only different disguises of the same Platonic level of being. [..] Different forms of the same mathematical concept or procedure are not only considered mathematically equivalent, but also historically equivalent.

---

[5] Andre Weil: *Who Betrayed Euclid? (Extract from a letter to the Editor),* Archive for the History of Exact Sciences 19 (1978), 91–93.

[6] Unguru S.: History of Ancient Mathematics: Some Reflections on the State of Art, Isis 70 (1979), 555–565.

An overview of the then dispute offers the overview article[7] by D.E. Rowe. Andre Weil here represents, in Rowe's opinion, the philosophy of the Bourbaki group. He is of the opinion that a little knowledge of group theory helps to make the content of Euclidean proportion theory (and other things in passing) understandable. His goal is quite different from that of listing the complex problems that occur in Books V and VII of Euclid's *Elements*. This mathematical block provides numerous, subtle difficulties for our understanding of Greek names for numbers, sizes and ratios and their mutual effects, difficulties that still puzzle experts today. Historians tend to ask themselves whether mathematical concepts always have a clear meaning—regardless of the cultural context in which they arise. Weil is not only convinced of this, but also of the opinion that he and other talents are able using modern algebra to solve the mysterious problems of the history of mathematics. Unguru calls this behavior *ahistorical:*

> If scholars continue to neglect the special, specific features of a mathematical epoch, whether due to explicitly given or tacitly accepted principles, then their work is ahistorical and should be labeled as such by the community of historians.

In response to Unguru, I. Bashmakova[8] spoke up, whose modern interpretation of Diophantus had been criticized several times. After a comparison of Chinese, Indian and Greek mathematics, she concludes together with van der Waerden that in *all* mentioned mathematics, the binomial formulas were always represented in the form of geometric surface transformations, in India and China without any connection to any geometric theorems. Only in Greece was geometry built on axioms and further developed, even when problems with incommensurability arose.

In particular, I. Grattan-Guinness[9] went hard on Bashmakova, claiming that she propagated the following two-step approach and thus contradicted Unguru:

> First, the [historical] text is translated into a contemporary mathematical expression; i.e. an equivalent model is created. This is absolutely necessary to develop the actual understanding of the text. In the next step, it is necessary to embed the work being considered in the mathematical context of its time.

The Bourbaki philosophy not only influenced the reception of the history of mathematics treated here, but also caused a considerable "modernization" of the curriculum in the seventies and eighties (cf. Fig. 1.1). He became especially well-known

[7] Rowe D.E.: New Trends and Old Images in the History of Mathematics, 1996; in the anthology Calinger.

[8] Bashmakova I.: A new view of the geometric algebra of the ancients, in Bashmakova.

[9] Grattan-Guinness I.: History or Heritage? An Important Distinction in Mathematics and for Mathematics Education, in Van Brummelen.

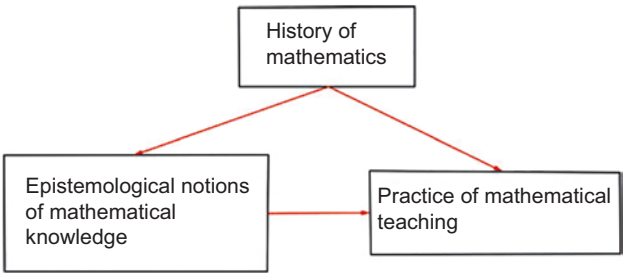

Fig. 1.1 Diagram of mathematics reception

for his lecture by J. Dieudonné[10], which was entitled: **Euclid must go!** and contained some radical hypotheses.

(101) This demand may be a shock to some, but I would like to show you some details with strong arguments in favor of this thesis. Let me first assure you that I have the deepest admiration for the achievements of Greek geometry. I consider their geometric inventions to be perhaps the most extraordinary intellectual achievement ever made by mankind. Thanks to the Greek spirit, we were able to overlook the towering construction of modern science.

(102) The basic concepts of geometry have been extensively analyzed to date, especially since the middle of the 19th century. This made it possible for us to create simple and robust foundations for the Euclidean work and thus to reformulate its significance in relation to modern mathematics; the foundations are separated from the unordered set of results that have no relevance except that they are scattered relics of inadequate methods or an outdated approach.

(103) The result may be a little disturbing. Let's assume—to simplify the argument—that the Euclidean geometry of the plane is to be taught to strangers from another world who have never heard of it or only want to have an insight into possible applications of modern research. Then I think the whole course could be tackled in 2 to 3 h—an hour will be needed with the description of the axiom system, another with useful consequences and the third possibly with some easy, interesting exercises.

(104) Everything else, which now fills whole volumes of elementary geometry and by that I mean, for example, everything about triangles (it is completely feasible and desirable to explain the whole theory without even defining a triangle!), almost everything about circle inversions, bundles of circles and conic sections etc., all of this has as much relevance for what (pure and applied) mathematics is today as magic squares or chess problems!

[10] Dieudonné J.: *New Thinking in School Mathematics, Organization for European economic cooperation, 1961.*

Supplement to the 2nd edition

You would hardly believe that after 45 years the discussion about the so-called "geometric algebra" is still going on; J. Høyrup[11] puts it more harshly: …*sometimes in disputes so hot that one would believe it to be blood.*

In his article on principles, Høyrup takes a neutral position between Unguru and the "older" authors, he checks whether the made quotes of the different authors are correct and the context was understood. In doing so, despite all criticism of Zeuthen, it turns out that nobody seriously read and understood his book on conic sections. He recalls the text of H. Freudenthal[12]:

> Whoever begins reading Greek mathematics is impressed by large parts that are obviously algebraic, as well as by other parts in which algebra apparently hides under a geometric shell. […]. S. Unguru has recently called this view into question. Everyone who has written about Greek mathematics has been wrong, he claims. For what reasons? Has he discovered sensational new facts? No, nothing! He has not even interpreted old facts in a new way. He simply says that they are wrong and does this with clear rhetorical emphasis. If the rhetoric is not taken into account, the rest consists of large excerpts from the work of others, which are provided with numerous exclamation and question marks, and some more precise statements that can be subjected to a proper analysis.

As Quote No. 64, he brings B. van der Waerden:

> Unguru denies the existence of a Babylonian algebra. Instead, he speaks, based on Abel Rey, of an arithmetic stage (of Egyptian and Babylonian mathematics) in which the argumentation proceeds largely elementary-arithmetically or is based on empirically paradigmatic rules that were derived from successful trials as a prototype. I have no idea what texts this statement is based on. For me, this is historiography at its worst: quoting the opinions of other authors and treating them as if they were established facts, without quoting the texts themselves. Let's stick to the facts and quote a cuneiform text BM 13901, which deals with the solution of quadratic equations. Problem 2 of this text reads: *I have subtracted the (page) of the square from the area, and 14.30 is it.* The statement of the problem is quite clear: It is not necessary to translate it into modern symbolism. If we translate it, we get the equation $x^2 - x = 14.30$.

Elsewhere, van der Waerden tries to relativize his then-contradiction:

> If this definition of "algebraic thinking" is accepted, Unguru is actually right when he concludes that "there was never any algebra in the pre-Christian era" and that Babylonian algebra never existed and that all claims of Tannery, Zeuthen, Neugebauer, and myself about "geometric algebra" are complete nonsense. Of course, this was in no way our definition of algebraic thinking. When I speak of Babylonian, Greek, or Arabic algebra, I mean algebra in the sense of Al-Khwarizmi or in the sense of Cardano's "Ars magna" or

[11] Høyrup J.: What is "geometric algebra" and what has it been in historiography? AIMS Mathematics, 2(1): pp. 128–160 (2017).

[12] Freudenthal H.: What Is Algebra and What Has It Been in History? *Archive for History of Exact Sciences* **16,** 189−200, 1977.

> in the sense of our school algebra. Algebra is therefore: The art of dealing with algebraic expressions like $(a+b)^2$ and solving equations like $x^2+ax=b$.

Høyrup provides a quote from D.E. Rowe[13], which demonstrates that Unguru's thesis is now generally accepted:

> Today, most historians of mathematics seem to have come around to accepting this central tenet [of Unguru's]. In fact, A. Jones told me at the symposium honoring Neugebauer at the Institute for the Study of the Ancient World (NY) that Unguru's position could now be considered accepted orthodoxy. Unguru, however, asks that we make a distinction; he quickly pointed me to the recent work of experts in Babylonian mathematics, which, in his opinion, continue to commit the same kinds of sins that he's *been complaining* about for so long.

The authors M. Sialoros and J. Christianides[14] have managed to find a valid definition of "What is the precursor of Algebra", now called *premodern Algebra*; Unguru has approved the manuscript. "Premodern" is recognized by a five-step approach:

- (1) All variables receive a name
- (2) All operations are only carried out with named variables
- (3) As a result, one or more equations are created
- (4) These equations are transformed and finally solved
- (5) The solution answers the question

They illustrate this with an example from al-Khwārizmī: *Sum of two numbers is 10, their product 21*

1. $A=x; B=10-x$
2. $x(10-x)=10x-x^2$
3. $10x-x^2=21$
4. $10\div 2=5; 5\times 5=25; 25-21=4; \sqrt{4}=2;\ 5+2=7;\ 5-2=3$
5. $A=7; B=3$ or vice versa

In an example from Abū Kāmils *Algebra* the distinction between "arithmetic" and "algebra" is to be shown: $x^2+10x=39$. Abū Kāmil calculates *arithmetically:* $\frac{10}{2}=5; 5^2=25; 39+25=64; \sqrt{64}=8; 8-5=3$. R. Rashed[15] explains the calculation process in a footnote using a quadratic complement:

$$x^2+10x=39 \Rightarrow (x+5)^2=64 \Rightarrow x=8-5=3$$

This is an *algebraic* procedure.

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[13] Rowe, D.E.: Otto Neugebauer and Richard Courant: On Exporting the Göttingen Approach to the History of Mathematics, *The Mathematical Intelligencer* **34:**2, 29–37, 2012.

[14] Sialaros M., Christianidis J.: Situating the Debate on "Geometrical Algebra" within the Framework of Premodern Algebra, Science in Context 29(2), pp. 129–150 (2016).

[15] Rashed R.: Abū Kāmil. Algèbre et analyse Diophantienne, de Gruyter 2012, p. 250.

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How Greek Science Began

2

For wonder is the feeling of a philosopher,
and philosophy begins in wonder.
(Plato: Theaitetos 155D)

Around 1400 BC, the powerful palaces of the Mycenaean-Minoan culture, such as Mycenae, Tirnys and Knossos, were destroyed and abandoned, the exact cause of which is unknown. The knowledge of the so-called linear script A was lost, around 1200 BC the linear script B was also forgotten. Around 1200 BC, there was a wave of immigration of Indo-European tribes of the Ionians, Achaeans and Dorians, probably from the Balkans. It is unclear whether a second wave of destruction (around 1200 BC), caused by the so-called Sea Peoples, is related to this, but a number of Anatolian sites (such as Troy) and island settlements (such as Crete) were destroyed. Knowledge of these attacks is conveyed by Egyptian papyri and temple wall paintings.

The historian *Thucydides* [I, 12] sets the two first waves of immigration at 60–80 years after the Trojan War, that is, around 1120 BC. The Dorians conquered a large part of the Greek mainland (still living in the Bronze Age) with iron weapons and founded Sparta. In doing so, they drove the Achaeans and Ionians living on the mainland to the Greek islands and the Anatolian coast, which was then inhabited by Persians, Lydians and Medes. Herodotus reports on this migration of peoples that the Ionians

> had founded their cities in an area that had the most pleasant climate in the entire known world.

Pausanias notes in his *Description of Greece:*

> The land of the Ionians enjoys the most favorable climate; it has sanctuaries such as one finds nowhere else. … The wonders in Ionia are numerous and hardly inferior to those in (the rest of) Greece.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_2

Fig. 2.1 Excerpts from Greek vase paintings

By about 700 BC, most of the city-states *(polis)* had been founded, which over time developed into economic and cultural centers. Initially, Sparta was leading in the Peloponnesian League. Inventiveness and craftsmanship, as well as the availability of slaves, allowed the Greeks to engage in shipbuilding, mining, metal processing, pottery and weaving. This was done so successfully that production exceeded self-consumption. Figure 2.1 shows, using vase paintings, the variety of Greek life: men at the smelter, in fist fighting, during the hunt and at feasts, women in livestock breeding, while playing music, during the lamentation of the dead, in child-rearing and while adorning the bride.

As a result, trade flourished in the Mediterranean, leading to the establishment of outposts and colonies on the coasts of the Black Sea, in southern Italy, and even in southern France. (Plato humorously said of this: *The Greeks sit around the Mediterranean like frogs around a pond.*) The English historian W.G. Forrest lists the following reasons for the social and intellectual upheaval of the seventh and sixth centuries:

- Expansion of trade and colonization in the Mediterranean
- The resulting increase in arts & crafts and agricultural production for export
- The awakening of philosophy, which dealt with the nature of things and demanded the freedom and individuality of the individual

In his work "From the Hekale of Callimachus" (1893), T. Gomperz writes:

> The colonies were the great experimental field of Hellenic spirit, on which it could test and develop the latent potential under the most diverse circumstances.

In addition to the colonies in the Mediterranean, important trading posts were also established in Persia, Egypt and North Africa. Competitors were primarily the Phoenicians, who lived in the area of present-day Lebanon and also colonized the Mediterranean. The Phoenicians' trading center was Tyros. Carthage, founded as a colony around 900 BC, later took over the Phoenician possessions and became so

powerful that it was only defeated by the Romans after three Punic Wars (end of 146 BC).

In addition to commercial profit, there was also a lively exchange of culture with the aforementioned peoples. For example, the Greeks took over the Phoenicians' most successful invention, the alphabet's written characters. In contrast to the Semitic-Arabic languages, which only write consonants, Greek shone through its vocalization. The poetry of *Homer,* which was created in the Ionian region in the eighth and seventh centuries BC, was initially only transmitted orally, but later, when it was recorded, it resulted in a unified language. At the Olympic Games, which took place irregularly from 776 onwards, there was a singing competition with the presentation of the *Iliad* and *Odyssey.*

It is estimated that of the population of Athens at the time of *Themistocles* (around 500 BC) only about half, in legal terms, were considered free; of these, only a small third had Athenian citizenship. Only this minority could exercise the right to vote and hold political office. As in the city-state, an upper class also emerged in the Greek colonies, which, due to their influence and wealth, no longer had to live off their manual labor. This privileged class had time and money to engage in art, culture, and philosophy.

Among the cities that profited most from trade was Miletus, which even had its own trading post in Egypt called Naukratis. Miletus, together with Chios, Ephesus, Samos, and others, formed the so-called *Ionian League.* Its most famous settlements in southern Italy (Magna Graeca) were Croton, Metapont, and Tarent, which later became the centers of Pythagorean activity. The sailors and traders of Miletus were thus able to acquire a wealth of knowledge in navigation, astronomy, ethnography, and geography. During the Persian Wars (500–479 BC), Miletus was initially spared, allowing it to continue to engage in lucrative trade. After the Ionian Revolt led by Miletus against the Persians, the city was completely destroyed in 496 BC. After its reconstruction, Miletus' dominance was broken, and it became subject to Athens in the First Athenian Maritime League (477 BC).

In the above-described favorable environment, a group of outstanding personalities appeared on the Ionian coast in the *Ionic Phase* (seventh to fifth centuries BC), whose world view was no longer determined by the traditional myths of the gods. Rather, they tried to give a comprehensive explanation of terrestrial and astronomical natural phenomena through rational thinking; this was the birth of natural philosophy, called *The Greek Miracle* in English literature. Why this event took place there and at that time has produced a number of comments that could fill entire libraries. The famous philosopher and mathematician B. Russell[1] begins chapter one of his *History of Western Philosophy* with the words:

> In all history, nothing is so surprising or so difficult to account for as the sudden rise of civilization in Greece. Much of what makes civilization had already existed for thousands of years in Egypt and in Mesopotamia, and had spread thence to neighbouring countries.

[1] Russell B.: Philosophie des Abendlandes, Piper 2004, p. 25.

> But certain elements had been lacking until the Greeks supplied them. What they achieved in art and literatureis familiar to everybody, but what they did in the purely intellectual-realm is even more exceptional. They invented mathematics and science and philosophy...

In his work *Vom Ursprung und Ziel der Geschichte* (1949), K. Jaspers assumes that there is an empirically definable period of time during which the basic categories of thought and the approaches of world religions emerged approximately simultaneously, in which people still live today. Jaspers calls this period of time, which he sets from about 800 to 200 BC, the *Axis Time of World History.* In this time, important cultural foundations and categories of thought were created independently of each other in China (Confucius and Lao Tzu), in India (Buddha, Upanishads), in the West (Homer, Thucydides) and in Palestine (Jewish prophets like Elijah, Isaiah), which are still relevant today and allow us to speak of a certain unity of world history. An *axis* of world history seems to be around 500 BC, in the spiritual process taking place between 800 and 200 BC. There lies the deepest cut in history; it is the man who emerged with whom we live today.

The (Greek) philosophers of this time are now referred to as *Pre-Socratics*. Among these wise men were Thales of Miletus and Pythagoras of Samos, who, according to tradition, founded the science of mathematics. Going beyond the Babylonians and Egyptians, who filled clay tablets and papyri with mere numerical examples, they tried to formulate principles and find general relationships.

After the end of the Persian Wars, Athens was able to take a leading position among the Greek city-states. During the time of Pericles (ca. 495–429), Athens was not only politically, militarily and economically leading, but it also formed a center of art and science. The Academy and the Acropolis were founded here; scientific teaching took place at the Academy of Plato or at the Lyceum of Aristotle at this time, therefore called the *Athenian Phase*. The position of power of Athens challenged Sparta. The Peloponnesian War, which was fought with the Spartans, weakened Athens so much that it came under Macedonian influence in 338.

After the death of Alexander (323 BC), the conquered empire was divided; the Ptolemies took over the rule in Egypt. They made Alexandria (one of the eleven cities of the same name founded by Alexander) the capital, which then became the center of trade and learning in the Mediterranean. The mathematicians *Euclid, Eratosthenes* and *Apollonius* worked here at the Museion and at the successor institution. Archimedes worked in Syracuse, but was in close contact with Alexandria. The period from 300 to 190, ending with the death of Apollonius, is called the *Alexandrian Phase*.

After the end of the Ptolemaic rule, Egypt became a Roman province, but for a long time it was still able to fulfill its leading role as a center of science. In the following four centuries, renowned mathematicians such as *Heron, Claudius Ptolemy, Pappus, Diophantus* and *Theon* with daughter *Hypatia* (latter around 415 AD) worked there. In Athens, the mathematician *Proclus Diadochus (successor)* is still to be mentioned, who, with his Euclidean and Platonic commentaries, provides valuable mathematical hints (around 480). This marks the end of the *phase of Hellenistic mathematics;* Antiquity ended with the collapse of the Roman Empire in 476 AD.

The *legacy of Hellenistic mathematics* was administered in Rome, Constantinople and Baghdad. Rome learned the arithmetic of the late Pythagorean *Nicomachus* in the Latin translation of Boethius. In Constantinople, *Leon, the geometer,* had access to all the works of Archimedes that were available at that time. In Rome, the land surveyors lived off Greek knowledge, in particular the teachings of Heron. In Baghdad, the caliphs had important works by Euclid, Archimedes, Apollonius and Ptolemy translated into Arabic and thus preserved valuable writings that were lost in the course of the centuries in their Greek originals. A large part of these Arabic writings was later translated into Latin in the Middle Ages or the Renaissance and thus made accessible to the scholarly world of Europe. This process is not yet complete; even today Arabic manuscripts can be found showing the contents of lost Greek writings.

2.1 The Origins of Mathematics

> When the Socratic philosopher Aristippus was thrown ashore on the island of Rhodes after a shipwreck and saw there geometrical figures drawn, he is said to have called out to his companions:
>
> *Let us be of good cheer, I see the traces of men*!
>
> (Marcus Vitruvius Pollio: *De Architectura*, Preface Book VI)

Mathematics originally developed as part of philosophy. Aristotle describes the beginning of philosophy as follows [Metaphysics 982B] (quoted from www.perseus.tufts.edu):

> It is through wonder that men now begin and originally began to philosophize;wondering in the first place at obvious perplexities, and then by gradual progression raising questions about the greater matters too, e.g. about the changes of the moon and of the sun, about the stars and about the origin of the universe. Now he who wonders and is perplexed feels that heis ignorant.
>
> Therefore if it was to escape ignorance that men studied philosophy, it is obvious that they pursued science for the sake of knowledge, and not for any practical utility. The actual course of events bears witness to this; for speculation of this kind began with a view to recreation and pastime, at a time when practically all the necessities of life were already supplied.

He later comes to speak of mathematics:

> All begin, as we have said, by wondering that things should be as they are, e.g. with regard to marionettes, or the solstices, or the incommensurability of the diagonal of a square;.... for a geometrician would wonder at nothing so much as if the diagonal were to become measurable.

Mathematics originally meant *That which one must learn* (μἄθεσις or μἄθεμα); the root is μανθάνειν (to learn). Plato still uses the term [Timaios 88c] partly in its original sense. Among the Pythagoreans and later among Aristotle, the word

mathematics takes on the special meaning it has today. A quote from *Anatolius* is preserved in the definitions of Heron[2]:

> Why is this subject called mathematics? The Peripatetics [students of Aristotle] say that while one can practice rhetoric, poetry, and popular music without having studied them, no one can understand what is called mathematics without having first studied it. This is why they say that the theory of these subjects is called mathematics.

The meaning of the word μάθημα develops from 'learning' in general more and more toward mathematical learning. The Pythagorean Archytas (ca. 350 BC) already used it to denote something like 'exact, mathematical science' (D14 LM = B1 DK):It seems to me that those studying the μαθήματα discern well, and it is by no means strange that they are able to think correctly about each thing. They handed down to us clearin sight about the speed of the heavenly bodies and their rising and setting, equally about geometry, numbers, and not least about music. For these μαθήματα seem to be siblings of one another.'So the mathematical sciences that will become the quadrivium (= geometry, arithmetic, astronomy, music) are here already seen in close kinship and were called μαθήματα. In Aristotle μαθήματα is exclusively used for mathematical learning, alongside μαθηματική. Aulus Gellius writes that (Noctes atticae I.9.6):'the older Greeks called geometry, the art of making sundials, music, and the other similar higher disciplines μαθήματα'. Aristotle reports [Metaph. 981 B] on the origin of mathematics:

> And as more and more arts were discovered, some relating to the necessities and some to the pastimes of life, the inventors of the latter were always considered wiser than those of the former.... Hence when all the discoveries of this kind were fully developed, the sciences which relate neither to pleasure nor yet to the necessities of life were invented, and first in those places where men had leisure. Thus the mathematical sciences originated in the neighborhood of Egypt, because there the priestly class was allowed leisure.

Heron subdivides the mathematical sciences:

> Mathematics is a science which investigates the fathomable both by thought and by the senses, in order to determine that which falls within its province.[…] The nobler and higher part consists of two main divisions, arithmetic and geometry, while the part concerned with the senses comprises six: reckoning, mensuration, optics, musical theory, (theoretical) mechanics, and astronomy. Neither tactics, nor architecture, nor popular music, nor calendar reckoning are parts of mathematics, nor is (practical) mechanics.

Greek mathematics was primarily geometry, systematically arranged by Euclid and extended to conic sections by Apollonius. The Pythagoreans used arithmetic primarily as a key to explaining the world. A precursor to the area later known as algebra or number theory can be found in the work of Diophantus. Arithmetic in the sense of the Pythagoreans is pursued by Nicomachus of Gerasa. Numerical

[2] Heroninis Alexandrini opera quae supersunt omnia, Buch IV, Teubner 1914.

calculations on a large scale are carried out in the Babylonian sexagesimal system, in the root calculation of Heron of Alexandria, and in the calculation of the table of chords by Claudius Ptolemy. Heron reports on geometry in the preface to his work *Geometrica:*

> The origins of geometry can be shown using philosophy. So that we do not violate the principles, it is appropriate to explain the definition of geometry. Geometry is therefore the science of figures and sizes and their changes and its purpose is to achieve them; the method of its representation is synthetic: it begins with the point, which is without extension, and goes through the line and surface into space.
>
> Geometry achieves its representation through abstraction; it first deals with the physical body and its material content; by removing the materiality it receives the mathematical body, which is spatial. By continued abstraction, it reaches the point again.

Heron describes the development of geometry in the preface to his *Metrica*[3]:

> In its beginnings, geometry, as the ancients teach us, was concerned with land surveying and land division, from which it was named geometry. Since this business was useful for people, its concept was extended so that the handling of measurements and divisions also progressed to solid bodies, and since the first theorems were not enough, those operations needed further research, so that even to the present moment much of it is still unresolved, although Archimedes and Eudoxos have treated the subject excellently.

An important quote comes from W. Schadewaldt[4], who differentiates the role of Greek mathematics in view of the Mesopotamian one:

> And yet experts in these matters [mathematics among the Babylonians and Egyptians] emphasize that mathematics takes a very different turn with the Greeks. Before, it was mostly about practical arithmetic for various purposes, land surveying or astrology. But now it is no longer pursued for practical interest, but because one is interested in the nature of the number itself, what one encounters in these considerations: the symmetry, on the one hand, in which one sees something divine, on the other hand, something foundational of being. This is the real turning point back then, away from the merely applicable to the principle. From here, the science of mathematics began and moved forward in strict problem development to this day. It is one of the most clear individual connections, where the progress of the problem movement from the Greeks over a period of time, where it then rested, is clearly visible up to today.

R. Wilder[5] gets to the point:

> The Babylonians developed mathematics to a level at which two fundamental ideas were ready to be discovered and developed: the concept of a theorem and the concept of a proof.

---

[3] Heronis Alexandrini opera quae supersunt omnia, Band III, Teubner 1914.

[4] Schadewaldt W.: Die Anfänge der Philosophie bei den Griechen, Suhrkamp Wissenschaft 1978, pp. 81–82.

[5] Wilder R.: Evolution of Mathematical Concepts, John Wiley 1968, p. 156.

2.2 The Greek Numeral Symbols

The Attic numeral representations

Since approximately the beginning of the fifth century BC, there has been the use of the Attic numbers, which are erroneously also called Herodian after the Byzantine author Herodian, who was the first to mention the numeral symbols. The initial letters of the number word are used to write the corresponding numerical value. This results in a system similar to the Roman numeral system developed almost at the same time. It uses (except for I = 1) the capital initial letter as a numeral symbol:

Pente (5) = Π; deka (10) = Δ; hekaton (100) = H; chilioi (1000) = X; myrioi (10000) = M

The intermediate values 50, 500, 5000 and 50000 are written by inserting the characters for 10, 100, 1000, 10000 in smaller form into the character "Π", which takes on the form of "Γ". The characters are written additively next to each other (see Fig. 2.2).

This script appeared in Attica on public display boards and the famous Salamis calculation table (second century BC). Figure 2.3 shows the board made of white marble, a tracing with counting stones and the labeling.

Here it applies: 1 talent = 6000 drachmas, 1 drachma = 6 obols. The Attic number signs can also be seen on the famous Darius vase (Naples), which shows a bearded Persian treasurer of Darius' war council making a calculation, in his hand a folding wax tablet with notes. From left and right, two envoys in Asian dress approach, paying tribute in the form of money and goods (Fig. 2.4).

The Milesian number representations

Using the alphabet as numerical symbols, (according to H. Haarmann[6]) was a Phoenician idea that the Greeks developed and many Semitic languages followed, such as Hebrew (Fig. 2.5). Each of the 22 letters there has a sound and a numeric value.

By expanding the Phoenician alphabet, the Greeks were able to represent more numbers. By inserting three extra symbols Ϛ (Digamma = 6), ϟ (Koppa = 90) and ϡ (Sampi = 900), they reached the representation up to 900. These "digits",

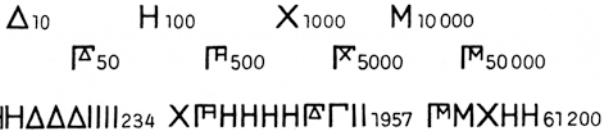

Fig. 2.2 Attic numerals

[6] Haarmann H.: Weltgeschichte der Zahlen, C.H. Beck 2008, p. 101.

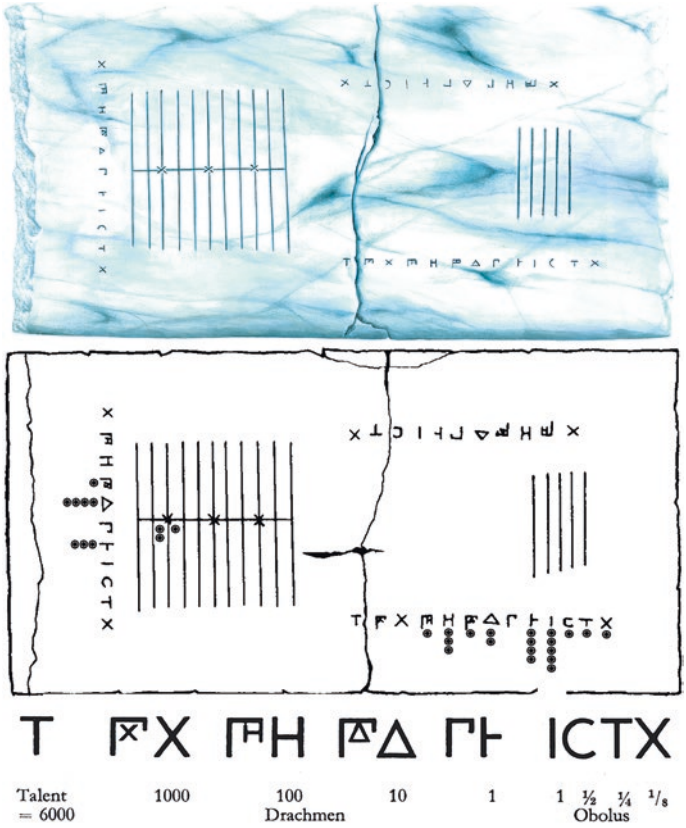

Fig. 2.3 Calculation table of Salamis and redrawing, science-illu

Fig. 2.4 Persian treasurer on the Darius vase (Wikimedia Commons)

named after the city of Miletus the Milesian numbers, became popular in the first century AD; Archimedes and Diophantus calculated with them (Fig. 2.6). The labeling of geometric figures using the letters A, B, Γ, Δ, E, etc. had the side effect of also being a numbering.

Hebräische Buchstaben	Namen und Umschrift der Buchstaben		Zahlwerte
א	ALEF	'	1
ב	BET	b	2
ג	GIMEL	g	3
ד	DALET	d	4
ה	HE	h	5
ו	WAW	v	6
ז	SAJIN	z	7
ח	CHET	ch	8
ט	TET	ṭ	9
י	JOD	y	10
כ	KAF	k	20
ל	LAMED	l	30
מ	MEM	m	40
נ	NUN	n	50
ס	SAMECH	s	60
ע	AJIN	'	70
פ	PE	p	80
צ	ZADE	s	90
ק	KOF	k	100
ר	RESCH	r	200
ש	SCHIN	sch	300
ת	TAW	t	400

Fig. 2.5 Hebrew numerical symbols

EINER				ZEHNER				HUNDERTER			
Α	α	Alpha	1	Ι	ι	Iota	10	Ρ	ρ	Rho	100
Β	β	Beta	2	Κ	κ	Kappa	20	Σ	σ	Sigma	200
Γ	γ	Gamma	3	Λ	λ	Lambda	30	Τ	τ	Tau	300
Δ	δ	Delta	4	Μ	μ	My	40	Υ	υ	Ypsilon	400
Ε	ε	Epsilon	5	Ν	ν	Ny	50	Φ	φ	Phi	500
Ϝ	ϛ	Digamma	6	Ξ	ξ	Xi	60	Χ	χ	Chi	600
Ζ	ζ	Zeta	7	Ο	ο	Omikron	70	Ψ	ψ	Psi	700
Η	η	Eta	8	Π	π	Pi	80	Ω	ω	Omega	800
Θ	θ	Theta	9	Ϙ	ϙ	Koppa	90	Ϡ	ϡ	San	900

Fig. 2.6 Milesian number signs

In order to expand the number range, the thousands were marked with single digits by a preceding underscore, e.g. 5000 = ,ε. Larger numbers like myriad were marked in later manuscripts with colons; previously, one had set the multiple over the "M". Diophantus marks the myriad by means of a following dot, an example is:

$$1\ 507\ 284 = \rho\nu.,\zeta\sigma\pi\delta$$

Here is a simple multiplication

$$25 \cdot 43 = 800 + 60 + 200 + 15 = 1075$$

$$\kappa\varepsilon \cdot \mu\gamma = \omega + \sigma + \xi + \iota\varepsilon =, \alpha o\varepsilon$$

The fact that a myriad was the largest number named caused Archimedes considerable difficulty in his treatise *The Sand Reckoning,* in which he estimated the number of grains of sand in the universe.

Fowler provides a comprehensive treatment of Greek arithmetic in Sect. 7.3 of his book; in Sect. 7.4 he discusses some well-known texts, such as "On the Measurement of the Circle" (Archimedes), "The Size and Distances of the Sun and Moon" (Aristarchus), the Papyrus London ii265, and the Ostrakon Bodleian ii1847.

2.3 The Greek School

At about the age of seven, the children, mostly boys, were taught by a teacher whose salary was financed by the parents of the pupils. The teacher was therefore dependent on the payments of the parents, which was not exactly conducive to his reputation. One example is the philosopher Epicurus, who was accused of being merely an elementary school teacher. From the third century BC, school education was better supervised, which led to the establishment of municipal or state schools. The schools were run by experienced school directors; around them were the parents and supporters who paid school fees and had the right to vote on the school management. Lectures and discussions were partly held in public. The school principal was responsible for the order in the school, held examinations and was authorized to give instructions to the teachers.

A speaker of the fourth century describes the legal provisions on school education, which go back to Solon, as follows (after Gigon[7]):

> Although the teachers to whom we must entrust our children earn their livelihood by their decency and are exposed to misery in the event of contrary behavior, it is nevertheless observed that the legislator does not trust them enough and thus makes very precise regulations first of all, at what time the free-born boy has to be in school, then how large the class may be and when the pupil can leave the school again. It is prescribed both for the teachers at the school and for the sports teachers in the *gymnasiums* that they may not open their business before sunrise and must close it before sunset, since loneliness

[7] Gigon O.: The culture of the Greeks, VMA-Verlag Wiesbaden 1979.

Fig. 2.7 Representation of the Paidagogos on Greek vases

> and darkness arouse the most serious fears … Finally, regulations are issued on the slaves (Greek παιδᾰγωγός), who accompany the children on the way to school, at the school festivals of the Muses, at the Hermes festivals on the sports grounds and when the children take part in choir events.

"For the moderation of young men" a separate supervision was installed by the Areopagus [Pseudo-Plato, Axiochos 366D]. The already mentioned *Paidagogos,* a slave acting as a companion, is first attested for the year 480 BC [Herodotus VIII, 75]. He enjoys only a low reputation [Plato, Alkibiades I, 122B]. Vase paintings and terracottas show him as a foreigner with a bald head, scraggly beard and cane (Fig. 2.7a, b). Constantly at the side of the child, he protected him/her from dangers and taught proper behavior and civilized conduct; some pedagogues may also have supervised the school tasks.

On the recommentation of Plato the curriculum is extended in Laws [Nomoi 819A-C], as O. Toeplitz[8] mentions:

> The Egyptian priests are said to have specially drilled their scholars in arithmetic and geometry — partly with a view to their use in land-mensuration. First, as regards counting, lessons have been invented for the merest infants to learn, by way of play and fun, — modes of dividing up apples and chaplets, so that the same totals are adjusted to larger and smaller groups, and modes of sorting out boxers and wrestlers, in byes and pairs, taking them alternately or consecutively, in their natural order. Moreover, by way of play, the teachers mix together bowls made of gold, bronze, silver and the like, and others distribute them, as I said, by groups of a single kind, adapting the rules of elementary arithmetic to play; and thus they are of service to the pupils for their future tasks of drilling, leading and marching armies, or of household management, and they render them both more helpful in every way to themselves and more alert.

[8] Toeplitz O.: Mathematik und Ideenlehre bei Platon, p. 55 in the collected volume Becker.

Even readings of poetry, rhetoric, philosophy and attention to newer sciences should make the youth useful and successful citizens; however, the youth are exhorted to moderation and virtue [Politeia 431A] [Nicom. Ethik X, 10, 8].

Relatively little is known of school books. E. Ziebarth[9] has collected many short fragments from school texts. The most extensive school papyrus from Hellenistic Egypt (Faiyum) was acquired in 1935 by two French researchers[10] in Cairo and published later; they called it "Book for the little student". Here is a short description of the papyrus by Gerson Schade[11]:

> The text begins with a writing exercise, in which all letter pairs (consonant, vowel) are practiced. This is followed by several number series, then a list of god and river names. This is followed by a small poetic anthology with excerpts from Euripides, Homer (Odyssey) and from an unknown comedy. This is followed by a list of square numbers up to 800 and a table of conversions of monetary units.

Other school papyri contain aphorisms and parables by Menander and Hesiod as well as philosophical sentences. There are also lists of deities, city inventories, lists of festivals or month names, weekdays and river or bird names.

At the age of 16–17, many young men were taken into the household of a wealthy older man who provided for their maintenance and education, including sexual matters; pederasty is attested in numerous vase paintings.

A young man aged 18–20 was called an *ephebe*, marking the beginning of his adulthood. At this age, the sons of the nobility and wealthy began to receive instruction in sports, which was socially promoted as it led to a pre-military education. Military service—initially only for the wealthy—was completed after graduation from school, first for a year as a recruit, then on active duty. Later, boys from families without full citizenship were also accepted, called *metoeci*. For them, completing military service was a prerequisite for later obtaining full citizenship.

The name Gymnasium (Greek γυμνάσιον) is derived from the Greek word for naked *(gymnos)*; exercises and sports were practiced by male nude athletes. The earliest examples of sports facilities date back to the sixth century BC; they were initially just fortified sand tracks, surrounded by trees, located near a sanctuary such as Delphi, Olympia and Nemea; proximity to a river or spring was required.

Some philosophical schools used the proximity of public sports facilities to build there teaching facilities, like the Academy and the Lykeion. There were also public lectures and discussions held, which were often also conducted in the attached libraries.

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[9] Ziebart E.: Aus der alten Schule, Maraus & Webers-Verlag Bonn 1910.

[10] Guéraud O., Jouguet P.: Un livre d'écolier du III^{eme} siècle avant J.-C., Publications de la société royale égyptienne de papyrologie 2, Cairo 1938.

[11] Schade G.: Pegasus-Onlinezeitschrift IV/3 (2004), p. 60 ff.

The administrators of the gymnasiums were called *Gymnasiarche*. A stele from Beroia (second century BC) describes the function of the Gymnasiarch in more detail:

> He was generally responsible for the administration and accounting and also had the power to impose fines or even physical punishment for gross violations of the house rules.

The stele further describes who was allowed to participate in sports: free men up to 30 years, but not slaves, freedmen, traders, male prostitutes, drunkards, madmen and physically disabled.

The Gymnasiarchs were elected annually and were required to organize and supervise the public games and competitions. This meant that they also had to take care of the fitness, morality and medical care of the athletes, which was associated with high costs.

The rhetorician Prodicus of Keos (c. 430 BC) describes the training phase as a long ordeal:

> The seven-year-old child, who has already experienced many hardships, now faces its first oppressors in the form of tutors, writing and physical education teachers. As the child gets older, they are called literature, geometry and drill teachers, again a multitude of slave drivers. If the child enrolls in the ephebes, it has to deal with military trainers and fear of beatings; then come the Lyceum and the Academy, as well as the gymnasion supervisors armed with rods—another bundle of evils.

Nothing is known about schools for girls. Only the famous poet Sappho is known to have gathered a group of students around her for instruction. Girls were educated in the family circle; education was committed to the ideal of beauty and virtue. The musical education includes quoting poems, playing an instrument and singing in a women's choir. The occupation in the household includes above all weaving (think of Penelope's long weaving of a carpet while waiting for Odysseus). At the age of 17–18, young women were married, the marriage contract being negotiated by the fathers. Plato also recommended this age as the time of marriage for women, 37 years for men.

Several teaching scenes are shown on the well-known *school vase* (Berlin F 2285) by the painter Duris/Douris (around 480 BC). Figure 2.8 depicts the music teacher with the lyre on the left, opposite him sits a student who accompanies him on a lyre. The center of the picture shows the literature teacher, who is holding a scroll; the text begins with "O Muse, find me on the shores of the Scamander the beginning of my song". The student opposite him recites the text from memory. On the right side sits the school principal *(didaskalos),* who supervises the lesson, as in the second picture.

On the other side of the shell, the music lesson can be seen on the left, with the student standing and listening to the flute (*aulos*) of the teacher (Fig. 2.9). In the middle, the teacher is checking the text of a student on an open writing board, holding a pen *(diptychon)* ready for corrections. The Greek term for music (Greek μουσική) not only includes playing music, but also speaking rhythmically with accompaniment and dancing.

Fig. 2.8 Teaching scene 1 (Wikimedia Commons)

Fig. 2.9 Lesson scene 2 (Wikimedia Commons)

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Thales of Miletus

3

3.1 Mathematical Work

Thales of Miletus (Θαλῆς ὁ Μιλήσιος) (640–546 BC) is the first philosopher and mathematician known by name (Fig. 3.1). According to Diogenes Laertius [I, 22], he was a merchant, so he also came to Egypt on his business trips, where he learned Egyptian mathematics from the priests (Fig. 3.2). According to Plutarch, he soon surpassed his teachers and, to the surprise of Pharaoh Ahmose II (570–526 BC), Greek *Amasis,* he determined the height of the pyramids using a stick (*gnomon*) by comparing the lengths of the shadows (Fig. 3.3). Plutarch writes in a text about Ahmose:

> Although he admires you [Thales] for other things as well, he esteems the measurement of the pyramids above all else, namely that you, without any trouble and without using an instrument, by sticking a rod into the end point of the shadow that the pyramid casts, from the two triangles that arise from the touch of the sunbeam, show that the one shadow has the same ratio to the other as the pyramid has to the rod.

He describes the same process in his work "On the Seven Sages" (Conv. Sept. Sap. 2147A). Pliny gives a similar account of it in his *Natural History*. An (anonymous) epigram of the Greek Anthology (IX,366) counts Thales among the Seven Sages: I will tell you in verse the cities, names, and sayings of the seven sages. Cleobulus of Lindus said *That measure was best*; Chilon in hollow Lacedaemon said *Know thyself*; and Periander, who dwelt in Corinth, *Master anger*; Pittacus, who was from Mytilene, said *Naught in excess*; and Solon, in holy Athens, *Look at the end of life*; Bias of Priene declared *that most men are evil*, and Thales of Miletus said *Shun suretyship*.

Thales is counted among the "Seven Sages" of Greece by all authors. He calls himself a friend (φίλος) of wisdom (σοφία). Proklos, who still knows the (later lost) history of mathematics of Eudemos of Rhodes (ca. 325 BC), attributes in his

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_3

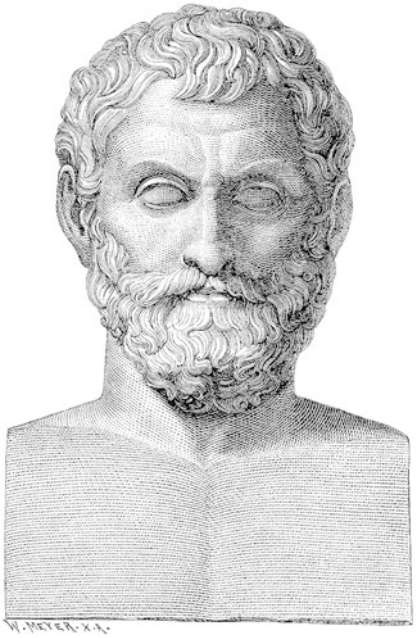

Fig. 3.1 Thales (Wikimedia Commons)

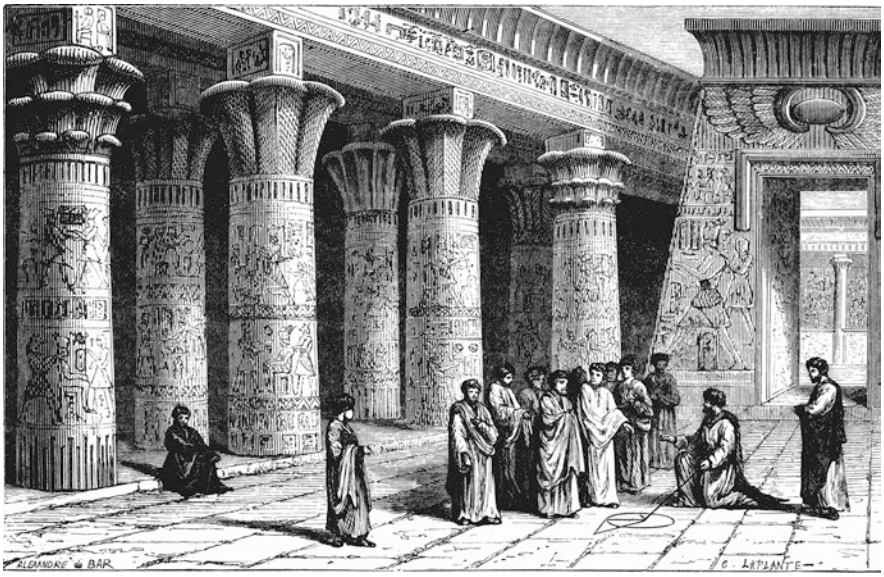

Fig. 3.2 Thales in front of the Egyptian priests, Göll 1876

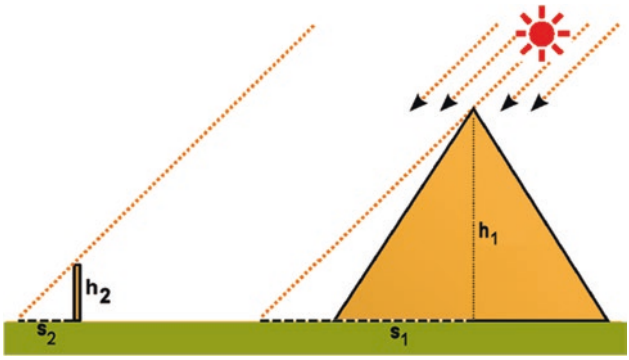

Fig. 3.3 Height measurement of the pyramid using a shadow stick

Commentary on the First Book of Euclid's Elements (around 450 AD) Thales the knowledge of the following principles:

- Opposite angles are congruent [Euclid I, 15].
- Base angles in an isosceles triangle are congruent [Euclid I, 5].
- A triangle is uniquely determined by one side and the two adjacent angles [Euclid I, 26].
- Circles are halved by their diameter [10th Axiom at Euclid]
- Corresponding sides of similar triangles are in the same ratio [Euclid VI, 4].
- The angle at the circumference in a semicircle is a right angle [Euclid III, 31].

After the definition of the right angle (R), the angle sum in a right-angled triangle can be determined. However, it must be assumed that the angle sum is constant in all these triangles.

Possible proof (Fig. 3.4): The right-angled triangle ΔABC with the altitude CD is considered. In the three (partial) triangles ΔABC, ΔADC, ΔDBC the following angle sums result:

$$S = \alpha + R + \gamma$$

$$S = \alpha + \beta + R$$

$$S = R + (R - \beta) + \gamma$$

Pairwise equating delivers

$$\beta = \gamma \quad \therefore \quad \alpha = R - \beta \Rightarrow \alpha + \beta = R$$

Inserting shows that the angle sum in the right-angled triangle is equal to two rights.

Given that Thales lived in Egypt for a long time, it is reasonable to assume that his mathematical knowledge reflects Egyptian knowledge. According to the Greeks, the origin of mathematics was in Egypt. Plato has Socrates say in [Phaidros 274C]:

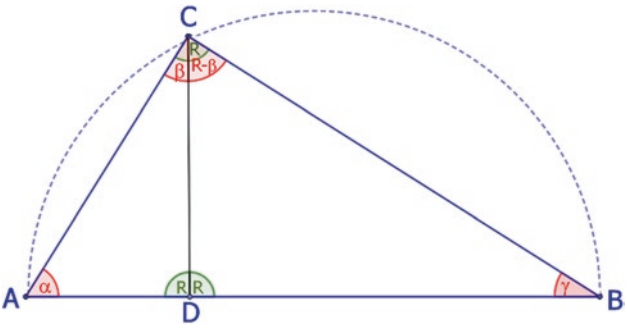

Fig. 3.4 To prove the angle sum

> I have thus heard that in Egypt, in the city of Naucratis, there was a certain ancient god, to whom also the holy bird, which they call the ibis, was dedicated; and this god's name was Theuth. Now it was Theuth who invented numbers and calculation and geometry and astronomy and dice and draughts, not to speak of checkers and backgammon.

Proclus writes about Thales' knowledge:

> Many things he himself discovered, and many things he handed down, beginning from first principles, to posterity; and both his own discoveries and those he transmitted he brought to greater perfection.

Thales is usually considered the founder of Greek geometry. Some authors believe that Thales had nothing to do with the theorem named after him, because the proof requires the knowledge of the sum of the interior angles of a triangle. This knowledge, according to Eudemus, only came to the Pythagoreans [Euclid I, 32]. T. Heath has found a way out; according to his view, the recognition of the point symmetry of a rectangle in a circle is sufficient for the proof.

D. R. Dicks expresses himself critically in his book "Early Greek astronomy to Aristotle" (1970, p. 43) ,

> It is probably to Eudemus that we owe the picture (enlarged by later tradition and dear to modem historians of science) of Thales as the founder of Greek mathematics and astronomy, the transmitter of ancient Egyptian and Babylonian science to Greece, and the first man to subject the empirical knowledge of the Orient to the rigorous, Euclidean type of mathematical reasoning. For this picture there is no good evidence whatsoever, and what we know of Egyptian and Babylonian mathematics makes it highly improbable.

Dicks judges previously (1959), *all the reports written centuries later are anecdotes of varying degrees of plausibility and historically worthless. This can be seen from the fact that not even the reports on the basic facts of his life agree. Generations of compilers had copied reports of their predecessors incorrectly and supplemented their biographies with fictitious attributes; thus, over time, a biographical tradition had emerged that had an apparent authenticity. It is also likely that the*

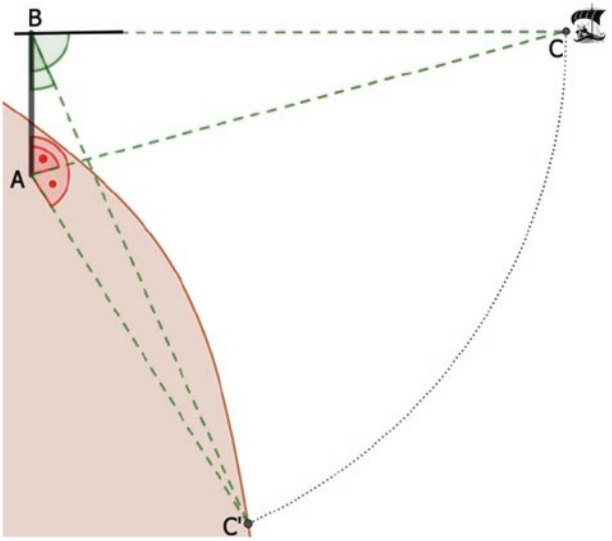

Fig. 3.5 Distance measurement by means of congruence theorem

philosophical positions attributed to him by Aristotle were in fact his own interpretations, which were later incorrectly attributed to Thales in the doxographic tradition. Dicks' criticism is certainly exaggerated, as the transmission of texts does not take place in a targeted manner, but is randomly determined and cannot be demanded by ancient authors to comment neutrally. This is a general problem of tradition. Later Dicks[1] reaffirms his views in a commentary on a book by S.Bochner. M. Cornelius[2] writes a reply to Dicks in a reprint from the University of Cologne. Reviel Netz insists (New History of Greek Mathematics, p.17):*Thales and Pythagoras did no mathematics whatsoever.*

Thales is said to have used the similarity theorem in determing the height of the pyramids described above. Likewise Thales is reported to have measured the distance of ships in the harbor of Miletus using only the congruence theorem according to Proclus.

One possible procedure is shown in Fig. 3.5: A rotatable rod AB, which has a horizontal direction indicator at the top, is set up vertically at the edge of the shore and rotated so that the pointer aims in the direction of the ship. Now turn the rod so that a point C 'can be targeted exactly over the pointer on the shore. The distance $|AC|$ of the ship can now be measured using the distance $|AC'|$, since the triangles ΔABC and ΔABC' are congruent (agreement in two angles and the side AB).

[1] D.R. Dicks: Commentary to Solomon Bochner, *The Role of Mathematics in the Rise of Science,* Princeton University 1966.

[2] www.rhm.uni-koeln.de/115/M-Cornelius.pdf [20.12.2013].

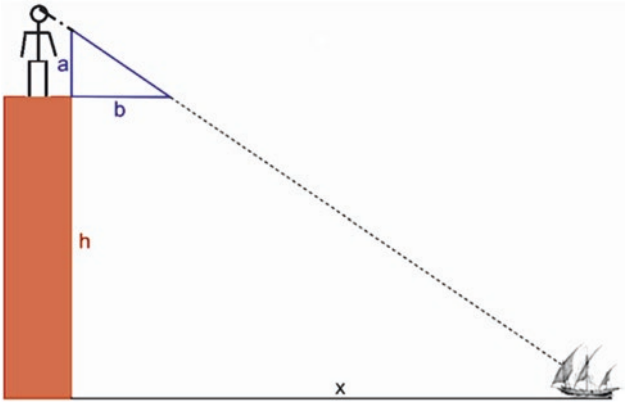

Fig. 3.6 Distance measurement using similarity

As Fig. 3.6 illustrates, the similarity theorem provides a much simpler solution for distance measurement. It is unclear whether Thales provided a proof for the stated theorems. Proclus attributes the proofs used in Euclid's *Elements* for the theorems mentioned above solely to Euclid.

There are no references of written works by Thales; even Plato or Aristotle lack any indication of a writing by Thales. Aristotle [Metaph. 983], however, reports on the School of Miletus which emerged from the teachings of Thales, that this school under the successors *Anaximander* or *Anaximenes* increasingly devoted itself to philosophy.

3.2 More Reports on Thales

Thales became famous during his lifetime for his prediction of the solar eclipse on May 28, 585 BC (Julian calendar). According to Herodotus [1, 74], this solar eclipse played a decisive role in the war between the Medes and the Lydians. O. Neugebauer vehemently denies that Thales himself, with his knowledge of Babylonian astronomy, would have been able to predict the date and, in particular, the visibility in Asia Minor. However, it is conceivable that Thales knew of the *possibility* of a solar eclipse. If he experienced the solar eclipse of 603 BC and knew the associated period (18 years + 11 days) of the Babylonians, he was able to *suspect* a new eclipse after this time. An (anonymous) epigram of the Greek Anthology (IX, 366) counts Thales among the *Seven Sages*: I will tell you in verse the cities, names, and sayings of the seven sages. Cleobulus of Lindus said "That measure was best"; Chilon in hollow Lacedaemon said "Know thyself"; and Periander, who dwelt in Corinth,"Master anger"; Pittacus, who was from Mytilene, said "Naught in excess"; and Solon, in holy Athens, "Look at the end of life"; Bias

of Priene declared “that most men are evil”, and Thales of Miletus said "Shun suretyship".

Eudemos quotes two (now lost) astronomical works *On the Equinoxes* and *On the Solstices,* which are attributed to Thales. The latter work is attributed to the Phoenician Phokos by Diogenes Laertios. H. Diels denies this attribution to Thales in his Fragments of the Presocratics. A. Szabó[3] on the other hand, is of the opinion that Thales had sufficient astronomical knowledge; since the solstices can be determined by means of a shadow rod. He also used a shadow rod (gnomon) to measure the height of the pyramid, as already mentioned. Timon of Phleius writes in his satirical poems *Silloi:*

> Thales alone of the Seven was wise in *astronomy*.

Theon of Smyrna reports that Thales was the first to give an explanation of a solar eclipse; This is also the content of a scholion to Plato Republic [600A]:

> Thales was the first to be called *sophos* and found that the sun is eclipsed when the moon passes in front of it.

The first mention of a gnomon is found in Herodotus (II, 109) as a transfer from Babylon. According to Eusebius, only Anaximander, who was probably a pupil of Thales, recognized the equinoxes. Similarly, Diogenes Laertius (2, 1) [DK 12 A1]:

> He [Anaximander] was the first to invent the gnomon and set it up in Sparta under “sundials”, as Favorinus reports in his “Colorful History”; this showed him the solstice and the equinoxes.

The anecdote of the Thracian maid, which Plato attributes to Socrates [Theaetetus 174a], is very popular:

> While he [Thales] was studying stars and looking upwards, he fell into a pit and a nice, witty Thracian maid jeered at him, they say, he was so eager to know the things in the sky that he could not see what was before him at his very feet.

In contrast, Aristotle [Politics 1259A] presents him as a viable businessman:

> All these methods are serviceable for those who value wealth-getting, for example the plan of Thales of Miletus, which is a device for the business of getting wealth, but which, though it is attributed to him because of his wisdom, is really of universal application. Thales, so the story goes, because of his poverty was taunted with the uselessness of philosophy; but from his knowledge of astronomy he had observed while it was still winter that there was going to be a large crop of olives, so he raised a small sum of money and paid round deposits for the whole of the olive-presses in Miletus and Chios, which he hired at a low rent as nobody was running him up; and when the season arrived, there was

[3] Szabó A.: Die Entfaltung der griechischen Mathematik, BI Wissenschaftsverlag 1994.

Fig. 3.7 Greek postage stamp for Thales (Wikimedia Commons)

> a sudden demand for a number of presses at the same time, and by letting them out on what terms he liked he realized a large sum of money, so proving that it is easy for philosophers to be rich if they choose, but this is not what they care about

Aristotle [Metaphysics 983B] writes about Thales' role in philosophy, that he discovered water as the primordial principle:

> Thales, the originator of this sort of philosophy, said that it was water (that is why he declared the earth to be sitting on water), perhaps drawing this supposition from seeing that the nourishment of all creatures is moist and that warmth itself arises from this and that it is by this that all creatures live (and the assumption that that from which a thing comes is its principle in all cases).

Aristotle here recalls Homer [Iliad 14, 201][14, 246], who tells how, in Greek mythology, Oceanus and Tethys [=sea gods] created the world of the gods.

In addition to astronomical observations, Thales is also said to have discovered the triboelectric effect of amber (Fig. 3.7). To explain this phenomenon, as well as magnetism, Thales attributes a soul to inanimate nature. The following fragment attributed to him can be found in Aristotle [De Anima 411a]:

> However, some say that [the soul] is mixed with the universe, which is perhaps why Thales believed that *everything is full of gods* [DK 11 A22].

Another fragment in Aristotle [de An. 405A] reads:

> According to what is reported, Thales also seems to have considered the soul as something movable, if he really said that the *magnetite is animated,* because it sets iron in motion [DK 11 A22].

Anecdote in the style of a *poeta doctus* is told by the poet Callimachus (Fr. 191):

Arkader Bathykles was dying and therefore arranged his possessions. He put a golden cup into the hands of Amphalkes, his middle son, with the order of giving it to the best of the Seven Wise Men.

> He sailed to Miletus: For the victory belonged to Thales,
> who was very intelligent overall and of whom it is said,
> he measured the stars of the [Little] Dipper,
> after which the Phoenicians sail.
> Now the Arkader found the old man scratching in the sand in the sanctuary
> of Apollo of Didyma, to draw the figure to which the Phrygian Euphorbos once came,
> who as the first man drew an scalene triangle
> *with a circle around it.*

The jambus of the poet explains Thales as the discoverer of the constellation "Little Dipper" and the rediscover of the "Theorem of Thales" after Euphorbos [= Pythagoras].

In English literature[4] the similarity theorem for triangles is also called "Theorem of Thales".

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Pythagoras and the Pythagoreans

4

The concept of numbers was born from superstition and hidden in the mysterious; … Numbers formed the basis of religion and philosophy and the artful figures had a magical effect on gullible people. (F. W. Parker)

In fact, everything that is recognized has a number. For it is impossible to grasp or comprehend anything without it. (Plato [DK 44 B 4])

4.1 Pythagoras of Samos

There is hardly any historical figure from antiquity whose biography is as controversial as Pythagoras'. Figure 4.1 shows a historical bust of Pythagoras (in an Oriental style) from Pompeii. For some, he was a philosopher, whose teachings found a multitude of followers even centuries later. For others, he was a religious sect leader, who founded a secret society teaching on vegetarianism and reincarnation, the workings of which were incomprehensible due to the formation of legends. W. Burkert sees him even more extremely, calling him a *shaman*.

Proclus takes over the following remark from the Eudemus report:

> According to this, Pythagoras transformed the occupation with this branch of knowledge into a real science by considering its basis from a higher perspective and investigating its theorems immaterially and intellectually. He is also the one who invented the theory of the irrational and the construction of the cosmic [Platonic] bodies.

The most important biographers Diogenes Laertios, Porphyrios and Iamblichos lived up to eight centuries later! We will limit ourselves here to mathematical content. Since Pythagoras left no written work, Aristotle could already no longer distinguish between his work and that of his followers. He writes in his lost book *On the Pythagoreans* (Apollonios *Histor. Mirab. 6*):

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_4

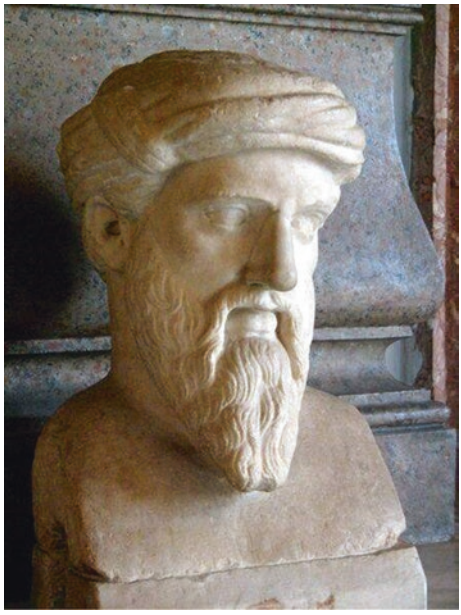

Fig. 4.1 "Capitoline" Pythagoras (Wikimedia Commons)

He initially worked in the field of mathematics and suddenly descended to the charlatanry of a *Pherekydes*.

Pythagoras was born around 570 BC as the son of a merchant family on the island of Samos off the coast of the flourishing trading city of Miletus. The Greek postage stamp of Fig. 4.2 shows a coin minted during the Roman period, which depicts Pythagoras sitting. In his left hand he holds the staff that identifies him as a scholar, his right hand holds a torch touching a world globe resting on a column.

Fig. 4.2 Greek postage stamp with Pythagoras coin (Wikimedia Commons)

In his youth he made numerous trips to the Greek colonies and came, probably for a long time, also to Egypt. Miletus had close relations with Egypt and had its own trading post there. Samos was a technologically advanced city. This is best seen in the famous 1036 m long tunnel of Eupalinos of Megara, which the tyrant Polykrates had drilled through the rock from both sides for the purpose of a water pipe. At the meeting place of the two drillings, this tunnel only has a vertical deviation of 60 cm. It was visited about 100 years later by the historian Herodotus (482–424 BC); in his *Histories* (III, 60) he has informed posterity about it.

After returning to Samos, Pythagoras taught the son of the tyrant Polycrates, but then left his hometown at the age of about 40 and settled in the city of Croton. Croton, Sybaris and Metapont were the most important settlements of the Greek colonies in southern Italy, which had existed since 700 BC; this area was later called *Magna Graecia* by the Romans. In four sensational speeches, he called on the population to lead a life based on morality, virtue, loyalty and worship of the gods. In doing so, he won a large following, which was connected to each other by living together and making promises of loyalty. Members of the group first went through a 5-year novitiate, were trained in arithmetic, music and astronomical observation, and had to lead a ascetic life (without meat and alcohol). The clothing was the white philosopher's robe, which could not be made of wool. The doctrine was determined by *vegetarianism*, the *catharsis* (spiritual purification through music), the *anamnesis* (recollection of pre-existence) and the *metempsychosis* (transmigration of souls) . Plato's comment [Gorgias 508] on the Pythagorean community is as follows:

> Yes, Callicles, wise men claim that partnership and friendship, orderliness, self-control, and justice hold together heaven and earth, and gods and men, and that is why they call this universe a world order, my friend, and not an undisciplined world-disorder.

Herodotus (IV, 95) calls Pythagoras *the most able philosopher among the Greeks*. W. Schadewaldt[1] evaluates Pythagoras' work neutrally:

> So Pythagoras made the beginning, at least he gave the impulse, even if we do not know in detail what he himself did. With him it begins that mathematics no longer appears as a means of mastering the world - this, too, soon reappearsand has played a major role up to the present day - but as pure mathematics,a way in which the phenomenon of relations, which are present in great abundance in the experiencableare present, can now be taken in itself and traced back to principles.

Iamblichos reports on the teachings of Pythagoras (XXIX):

> Afterwards, he teaches the whole of physics, and unfolds completely ethical philosophy and logic. He likewise delivers all-various disciplines, and the most excellent sciences.

[1] Schadewaldt W.: Die Anfänge der Philosophie bei den Griechen, Suhrkamp Wissenschaft 1978, p. 282.

> And in short there is nothing pertaining to human knowledge which is not accurately discussed in these writings. If therefore it is acknowledged, that of the [Pythagoric] writings which are now in circulation, some were written by Pythagoras himself, but others consist of what he was heard to say, and on this account are anonymous, but are referred to Pythagoras as their author; — if this be the case, it is evident that he was abundantly skilled in all wisdom.

Historical and philological research of previous centuries was inclined to believe the biographies of Porphyry, Iamblichus and other authors of Late Antiquity, in which Pythagoras was attributed with a whole range of significant discoveries in the fields of mathematics, astronomy and other sciences, in addition to numerous wonderful and supernatural deeds. Later, these witnesses were regarded, in accordance with the critical spirit of modern times, as a kind of myth-making that was fostered within the Neopythagorean and Neoplatonic schools.

The first investigation of the "Pythagorean question" came from E. Frank's essay on "Plato and the so-called Pythagoreans" (1923), which undertook a radical re-evaluation of the scientific knowledge gained so far. The Pythagorean discoveries in the field of mathematics and astronomy were, according to Frank, only made after 400 BC, that is, at the time of Plato, by Archytas and his school, and not without the essential influence of Democritus' atomism. We have no reason, according to the author, to speak of a Pythagorean science before this time.

W. Burkert's work on the Pythagoreans, which comes to the conclusion that the contribution of early Pythagoreanism to science was practically zero and that Greek mathematics must have had a non-Pythagorean origin, also belongs in the same direction.[2] Burkert shows (p. 409) that the main evidence for Pythagoras's mathematics does not go back to Aristotle and his pupils, but to other late sources, or it is influenced by the later Pythagoreans' occupation with mathematics. The earliest sources do not mention any achievements in this area. J. Høyrup said about this book, *Burkert had Pythagoras executed with it!*

M.L. Gemelli-Marciano[3] writes that Herodotus's warning that some Greeks had passed off Egyptian wisdom as their own knowledge was aimed at Pythagoras. She also mentions that Heraclitus called the wisdom, learning, and fraud of Pythagoras plagiarism from various sources. She continues:

> There is great uncertainty about these sources: Orphic poetry, Hesiod, Pherekydes, Anaximander?

Unfortunately, her source citation is not very precise. Perhaps she is referring to the following quotes: Heraclitus judges in [DK 22 B122]:

[2] Burkert W.: Weisheit und Wissenschaft, Hans Carl 1962, p. 202.

[3] Gemelli-Marciano M.L.: Die Vorsokratiker Band I, Artemis & Winkler 2007, p. 177.

> Pythagoras, son of Mnesarchus, has made the most investigations of all men and, after he has chosen these writings, he made for himself a wisdom out of them: much learning, deception.

Diogenes Laertius[4] (VIII, 6) refers to this quote:

> Some people mistakenly claim that Pythagoras left no writings, for Heraclitus, the natural philosopher, emphasizes quite loudly: *Pythagoras, son of Mnesarchus, has investigated more than any other man, has chosen these writings, and has fabricated a wisdom out of them: Know-it-all, cheat-it-all.*

The word γόητος is derived from the Greek word γόης (fraud, juggler). Burkert translates it with *shaman;* he found the word with Herodotus, who reports on shamans among the Scythians. The accusation of shamanism has been extensively justified by W. Burkert[5, 6]. It can be summarized as follows: His concern was not scientific, but speculative cosmology, number symbolism, and especially the use of magical techniques in the sense of shamanism. For his followers, he was a superhuman being and had access to infallible divine knowledge. The legitimation of this claim served the miracles attributed to him. Scientific efforts were only added later after the death of Pythagoras. There can be no talk of a Pythagorean mathematics during the lifetime of Pythagoras, but only from the time of the Pythagorean *Philolaos.* He thus makes Philolaos, who also developed the Pythagorean music theory, the ancestor of Greek mathematics.

In recent times, voices have been raised in opposition to this critical direction, which recognize in the testimonies of Iamblichus and other Neoplatonists information that goes back to a time when there were still survivors of the Pythagorean community. These reports could therefore contain information based on a real historical basis. In a series of recent work, material has been analyzed that has been completely ignored by philologists so far (e.g. numismatic data). It turned out that some of what was previously referred to as legend is confirmed by these data. But this insight did not lead to a change of heart by the Pythagoras critics. K. von Fritz[7] can be considered as representative of this mediating tendency, who have published a number of fundamental works on early Pythagorean science.

L. Zhmud[8,9] opposes Burkert's thesis. Zmud believes that there were no shamanistic phenomena typical of shamanism in the Greek-speaking cultural sphere

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[4] Diogenes Laertius, Jürß F. (ed.): Lives and Teachings of the Philosophers, Reclam 1998.

[5] Burkert W.: Weisheit und Wissenschaft: Studien zu Pythagoras, Philolaos und Platon, Hans Carl Verlag Nürnberg 1962.

[6] Burkert W.: Lore und Science in Ancient Pythagoreanism et al. 1972.

[7] von Fritz K.: Grundprobleme der Geschichte der antiken Wissenschaften, de Gruyter Berlin 1971.

[8] Zhmud l.: Philosophie und Religion im frühen Pythagorismus, Akademie Verlag 1997[1], de Gruyter 2016[2].

[9] Zhmud l.: Pythagoras und the Early Pythagoreans et al. 2012.

Fig. 4.3 Painting by Bronnikov: Pythagoreans celebrate the sunrise (1869), (Wikimedia Commons)

at the time of Pythagoras. He does not see any influence of Siberian or Oriental shamanism on Pythagoras (as Burkert does). In his opinion, the reports of Pythagoras' disciples' belief in their teacher's superhuman abilities and deeds are incredible. This legend-making about Pythagoras took place centuries later and can also be found in hagiographies of all religions. The historical Pythagoras was a philosopher who was interested in mathematics, music theory and astronomy, and his disciples carried out relevant research. Among other things, some theorems may go back to Pythagoras, which were later proven by Euclid.

4.2 The Pythagoreans

However, in the Greek colonies in southern Italy, a new philosophical school arose in the middle of the sixth century BC. This was the *Pythagorean Brotherhood,* named after its founder. Figure 4.3 shows a group of Pythagoreans (in white philosopher's robes) greeting the sunrise during their morning worship.

The origin of this school ultimately goes back to the Ionian cultural circle, because Pythagoras himself came from the Ionian island of Samos. After Pythagoras had spent a long time in Egypt, he did not return to his homeland, but settled in the southern Italian city of Croton, where he founded a brotherhood or sect (see Sect. 4.1). Its members committed themselves to following the "Pythagorean way of life", which consisted of a large number of rules for lifestyle, nutrition and asceticism; surely the Pythagorean doctrine of numbers was also taught.

In the following decades, the Pythagorean doctrine spread throughout the Mediterranean. Well-known Old Pythagoreans are Hippasos from Metapont (allegedly the discoverer of the irrational), Philolaos from Kroton and Archytas

from Tarent. Philolaos wrote a first natural philosophy, which Plato allegedly used as a source for his *Timaios* dialogue. Archytas was a student of Philolaos and the rescuer of Plato, because he equipped the ship that freed him from captivity on Sicily. Reviel Netz lets Greek mathematics begin with the "generation of Archytas", the basic knowledge he finds in Babylon (New History of Greek Mathematics, p. 53ff).

In the early days of the Pythagorean school, the religious-philosophical doctrine of Pythagoras had a purely esoteric character; it was not recorded in written form. For this reason, and because the Pythagoreans had the tradition of attributing all the achievements of the school to its founder, it is practically impossible to separate Pythagoras' own contribution from that of his students.

Such groupings of the elite aroused the suspicion of the common people. When a larger group from Sybaris once sought asylum in Kroton, Pythagoras advised against handing these people over. This led to a military confrontation in which Sybaris was defeated. When the war booty was distributed, there were political unrests which made it impossible for the Pythagoreans to stay. Pythagoras and his followers first fled to Tarent and finally to Metapont, where he died in 497 BC. His house there was long held in honor as a temple, so that Cicero could personally inspect it in 78 BC. Cicero speaks to Brutus and remembers [*De finibus* V, 2, 4]:

> You remember how I oncecame with you to Metapontum, and would not go to the house where we were tostay until I had seen the very place where Pythagoras breathed his last and theseat he sat in.

In fact, it can be assumed with high probability that the Pythagorean school was interested in mathematical problems from the time of its foundation and that the principle *Everything is a number* goes back to Pythagoras himself. Like other theories of early Greek authors, this principle represented a generalization from a very limited number of observations. Not only the ancient evidence, but also the early mathematical terminology indicate that these observations were connected with music. Of decisive importance was the discovery that the intervals of the tonal scale can be expressed in integer ratios: 1:2, 2:3 and 3:4. This discovery served as an incentive to search for similar ratios in other areas, e.g. in geometry and cosmology.

The meaning of the principle *Everything is a number* was the conviction that in every thing there are hidden numbers or number ratios. The task of knowledge is therefore to discover these ratios in a similar way as they had been found in music. In principle, this was about the numbers of the first decade. Some of these numbers were assigned a particularly important role, such as the three (τριάς), the four (τετάς), the seven (ἕβδομς) and the ten (δεκάς). The one was not considered a number at all; it was the starting point and the origin of all numbers and therefore of all things. Schematically, the range of numbers can be divided into:

Arithmetic	Numbers in themselves
Geometry	Numbers in space
Musical harmony	Numbers in time
Astronomy	Numbers in space and time

The search for simple number relationships could lead to scientific results in individual cases. It is quite possible that the Pythagoreans discovered that lines that are related like 3:4:5 form a right-angled triangle. This theorem, named after Pythagoras, was already known in Old Babylon. Many basic theorems of number theory arise from the clever use of pebbles (tokens). The formation of polygonal numbers, i.e. the formation of tokens in the shape of triangles, squares, rectangles, etc., was particularly popular.

Aristotle writes in his Metaphysics [I, 985B], that the Pythagoreans wanted to apply their principle from music also to the universe:

> At the same time, however, and even earlier the so-called Pythagoreans applied themselves to mathematics, and were the first to develop this science; and through studying it they came to believe that its principles are the principles of everything. And since numbers are by nature first among these principles, and they fancied that they could detect in numbers...many analogues of what is and comes into being... and since it seemed clear that all other things have their whole nature modelled upon numbers, and that numbers are the ultimate things in the whole physical universe, they assumed the elements of numbers to be the elements of everything, and the whole universe is one harmony and one number.

4.3 Mathematical Discoveries of the Pythagoreans

In addition to this philosophy of numbers, the Pythagoreans were occupied with the following mathematical discoveries.

1. They knew the theorem named after Pythagoras and its inverse. The side numbers (3; 4; 5) of the smallest right-angled triangle were generalized for the case where the hypotenuse exceeds a cathetus by one: $\left(2n+1; 2n^2+2n; 2n^2+2n+1\right)$, for example (5, 12, 13).
2. They knew the angle laws at parallels and used them to find out that the angle sum in a triangle is equal to two rights. They knew the theorem of the exterior angle and the interior angle sum of all regular polygons.
3. They invented the principle of area transformations, which were later called "geometric algebra". Comparings areas in three cases (equality, excess, deficiency) were used by Apollonius as a construction principle and naming for the conic sections.
4. They had a theory of proportions, which they wanted to apply to all similar figures. The theory initially only included rational ratios; the discovery of irrational ratios (in the case of the square diagonal or division of the pentagram) caused problems. The irrationality of $\sqrt{2}$ served Aristotle (thus before Euclid)

later as a prototype of a contradiction proof *(reductio ad absurdum)*. The attempt to treat roots and other irrationalities using double proportionality was then successfully continued by Euclid. The treatment of quadratic irrationalities in Book X of the *Elements* goes back to Theaetetus and certainly exceeds the Pythagorean knowledge.

5. They discovered the five regular polyhedra—later named after Plato—and were able to construct them completely. A large part of the constructions given later by Euclid in Book XIII are likely to go back to the Pythagoreans.
6. They laid the foundation of number theory with their definitions of even and odd, prime and composite numbers. The late Pythagoreans, such as Nicomachus and Iamblichus, also introduced concepts such as deficient and abundant, friendly and perfect numbers. Some numbers, such as four and ten, were given special symbolic meaning. The *tetractys (fourfoldness)* in the form of a triangular number $1+2+3+4=10$ was even the subject of the Pythagorean oath. According to Nicomachus "1" symbolizes a point, "2" a straight line, "3" a triangle and "4" a pyramid. He also considered (6; 8; 9; 12) to be another tetractys, since these four numbers are in a harmonious proportion (Fig. 4.4 after Nicomachus).
   About the (holy) number 10, Aristotle writes mockingly at the aforementioned passage:

   > Since they consider the number 10 to be something perfect and encompassing the whole nature of numbers, they also claim that the number of the moving heavenly bodies is 10; although only nine are actually visible.

7. They discovered important arithmetic formulas using the figurate numbers that they realized with pebbles (Ψηφοϛ), such as the sum of natural and odd numbers. An example is the decomposition of a square number into two consecutive triangular numbers (Fig. 4.5)

$$\frac{1}{2}n(n-1)+\frac{1}{2}n(n+1)=n^2$$

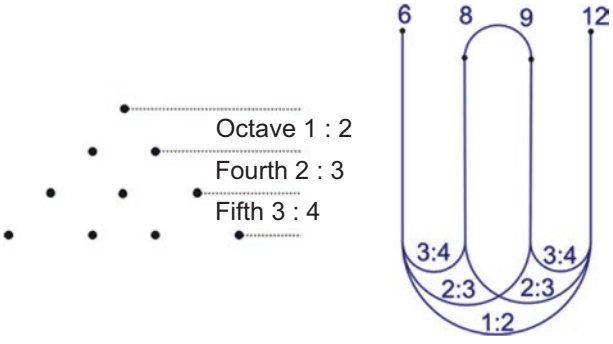

Fig. 4.4 Tetractys representations

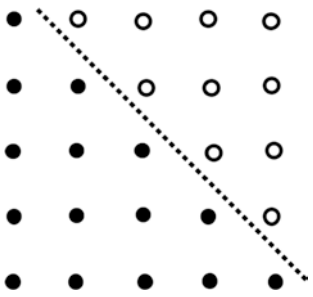

Fig. 4.5 Decomposition of a square number into 2 triangular numbers

8. They knew the three means (arithmetic, geometric and harmonic) and interpreted these mean formations in many ways, especially in music.
9. They knew that there are three regular polygons that, when placed next to each other in one point of the plane, cover its surroundings seamlessly. Such a tiling (parqueting) of the plane is provided by six equilateral triangles, three regular hexagons or four squares.

4.4 Figured Numbers

Many relationships over number sequences the Pythagoreans have won by laying down of pebbles. Aristotle speaks in his metaphysics [1092B] about it, *that certain people bring numbers into the shape of triangles and rectangles.*

Oblong Numbers
The partial sums of the even numbers are called rectangular numbers, Greek *Heteromeken* (Fig. 4.6)

$$\begin{aligned} 2 &= 1 \cdot 2 \\ 2+4 &= 2 \cdot 3 \\ 2+4+6 &= 3 \cdot 4 \\ 2+4+6+8 &= 4 \cdot 5 \\ 2+4+6+8+10 &= 5 \cdot 6 \end{aligned}$$

The n-th oblong number has the formula

$$2+4+6+8+\ldots+2n = n(n+1)$$

Fig. 4.6 Oblong (Rectangular) Numbers

Triangular numbers
The partial sums of the natural numbers were laid out in the form of triangles

$$\begin{aligned}
1 &= \tfrac{1}{2} \cdot 1 \cdot 2 \\
1 + 2 &= \tfrac{1}{2} \cdot 2 \cdot 3 \\
1 + 2 + 3 &= \tfrac{1}{2} \cdot 3 \cdot 4 \\
1 + 2 + 3 + 4 &= \tfrac{1}{2} \cdot 4 \cdot 5 \\
1 + 2 + 3 + 4 + 5 &= \tfrac{1}{2} \cdot 5 \cdot 6
\end{aligned}$$

Bracketing out the factor 2 from a sequence of oblong numbers, you get a sum of natural numbers. Thus, the n-th triangular number has the formula

$$1 + 2 + 3 + 4 + \ldots + n = \frac{1}{2} n(n+1) = \binom{n+1}{2}$$

Square numbers
The partial sums of the odd numbers were laid out in the form of squares

$$\begin{aligned}
1 &= 1^2 \\
1 + 3 &= 2^2 \\
1 + 3 + 5 + &= 3^2 \\
1 + 3 + 5 + 7 &= 4^2 \\
1 + 3 + 5 + 7 + 9 &= 5^2
\end{aligned}$$

Thus, for the sum of odd numbers follows

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

The sequence of square numbers can therefore be generated by continued addition of an odd number. Visually, a gnomon (angle hook) is placed on a square and thus generates to a new square (Fig. 4.7).

Pentagonal and hexagonal numbers
If regular pentagons and hexagons are laid out with pebbles, one speaks of pentagonal or hexagonal numbers (cf. Fig. 4.8).

The sum formula for the pentagonal numbers is:

$$1 + 4 + 7 + \cdots + (3n-2) = \frac{1}{2} n(3n-1)$$

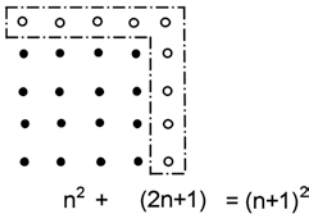

Fig. 4.7 Generation of square numbers by addition of odd numbers

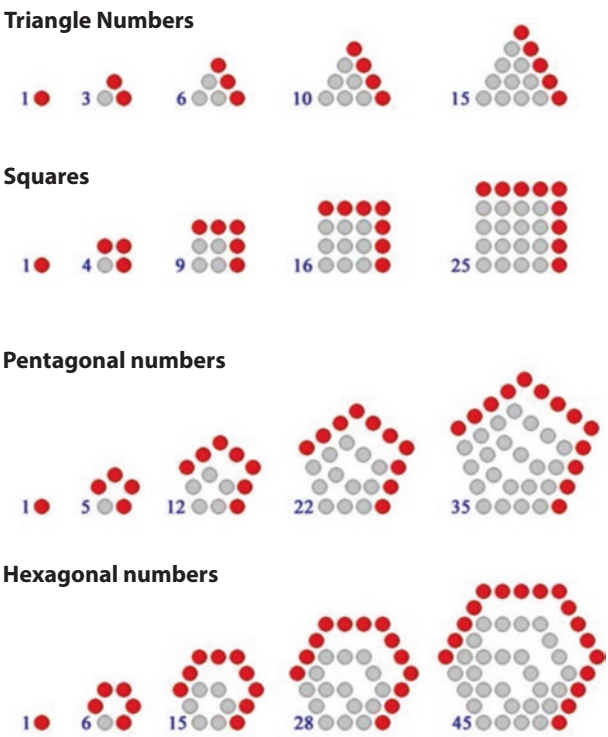

Fig. 4.8 Representation of the figurate numbers

Analogously for the hexagonal numbers:

$$1 + 5 + 9 + \cdots + (4n - 3) = 2n(n - 1)$$

If the figure underlying it has k corners, the general recursion formula for the n-th polygonal number $P(k, n)$ of the series

$$P(k, n) = \left(\frac{k}{2} - 1\right)n^2 - \left(\frac{k}{2} - 2\right)n$$

The formula does not apply to the oblong numbers, since $(k = 4)$ stands for the square numbers. For $(k = 4; n = 6)$, for example, the sixth square number is

$$P(4; 6) = \left(\frac{4}{2} - 1\right)6^2 - \left(\frac{4}{2} - 2\right)6 = 6^2$$

The representation of square and oblong numbers also provides an interpretation of the Aristotelian passage [Physics 203A10] about the Pythagoreans:

> The former [the Pythagoreans] also say that the unlimited (απειρον) is the even number. For by being enclosed and brought about by the odd number, it provides for the unlimited in the multitude of beings. A sign of this is what happens with numbers. If the gnomon is arranged around the one and without the one, in one case a different figure always results, in others only one.

This can be interpreted as follows:

The generation of square numbers from the one [unit] by successive addition of gnomons always yields squares; thus similar figures to each other. In contrast, the rectangle numbers are generated from the two by adding gnomons. However, the rectangles thus formed are *not* similar; thus always yield different figures (Fig. 4.9). This is an interpretation of the statement about the boundedness of the odd numbers compared to the unboundedness of the even numbers, which is nowhere explained in the extensive Pythagorean literature.

Plutarch mentions (in *Plat. Quaest.*) that four oblong numbers, placed around the one, result in a square figure. Since each oblong number can be decomposed into two triangle numbers, the following formula applies

$$8 \cdot \frac{1}{2} n(n+1) + 1 = 4n^2 + 4n + 1 = (2n+1)^2$$

Every odd square number ≥ 9 can therefore be decomposed into a sum of eight triangular numbers and one (Fig. 4.10). This property is used several times by Diophantus in his arithmetic. So in modern formulation

$$(2n+1)^2 \equiv 1 \; mod \; 8$$

Binomial formulas could also be laid out with pebbles (Fig. 4.11), like

$$(n+1)^2 = (n-1)^2 + 4n$$

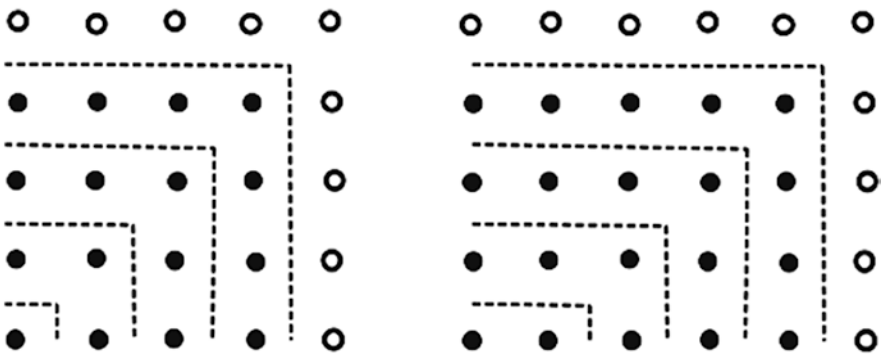

Fig. 4.9 Generation of square numbers and oblong numbers

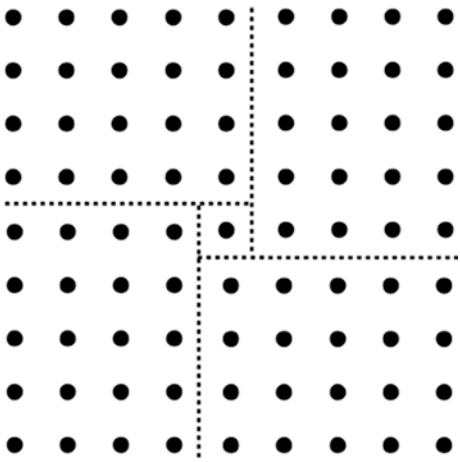

Fig. 4.10 Every square of an odd number is 1 mod 8

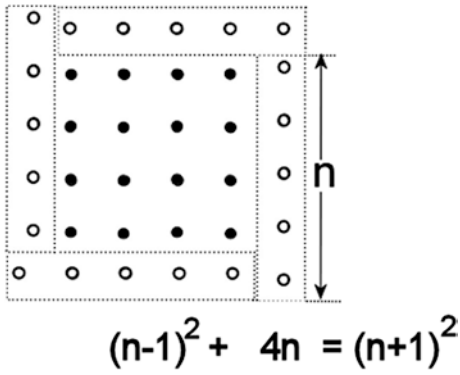

Fig. 4.11 Representation of a binomial formula

It should be mentioned at this point that there are critical voices that consider the figured numbers to be a later trivial invention of the Neo-Pythagoreans. For example, K. Reidemeister[10]:

> M. Cantor, T. Heath and others are more or less under the influence of Nikomachos and Theon, and I therefore cannot avoid the task of unmasking the allegedly archaic theories of means and figured numbers as Neo-Pythagorean *Pseudomathematics* and reducing them to their trivial mathematical core.

[10] Reidemeister K.: Das exakte Denken der Griechen, Claasen & Goverts 1949, p. 23.

4.5 Pythagoras' Theorem

The proof in [Euclid I, 47] is a congruence proof. The notation is taken from Fig. 4.12. It is carried out in five steps.

1. Claim: ΔABL is congruent to ΔAEC.
   Since the quadrilaterals $AEGB$ and $ACKL$ are squares, the sides of the triangle $|AE|$ and $|AB|$ or $|AL|$ and $|AC|$ match. The $\measuredangle EAC$ is equal to the sum of α and the interior angle 90°. Thus, $\measuredangle EAC$ is congruent to the $\measuredangle BAL = \alpha + 90°$. Thus, the two triangles match in two sides and the included angle and are congruent by the SAS criterion.
2. Claim: ΔAEC is equal in area to half the rectangle $AEFD$. The triangle has the rectangle side AE as its base and the rectangle side AD as its altitude. Thus, the area of the triangle is half the area of the rectangle.
3. Claim: ΔABL is equal in area to half the square $ACKL$. The triangle has the square side $|AL|$ as its base and the square side $|AC|$ as its altitude. Thus, the triangle's area is half the square's area. From Steps 2 and 3 it follows finally that the rectangle $AEFD$ is equal in area to the square $\square ACKL$.
4. In an analogous way, one proves that the triangles ΔABH and ΔCBG are congruent. Thus, as in Step 3, it follows that the rectangle $DFGB$ is equal in area to the square $\square BHIC$.
5. Equality of areas applies: $\mathcal{F}(AEGB) = \mathcal{F}(AEFD) + \mathcal{F}(DFGB) = \mathcal{F}(ACKL) + \mathcal{F}(BHIC)$. The square over the hypotenuse is equal in area to the sum of the squares of the catheti.

Step 3 is the congruence proof of the Pythagorean theorem of Euclid, but it only appears implicitly in the *Elements*. Euclid also knew the extension of the Pythagorean theorem to similar figures over the sides of the right-angled triangle

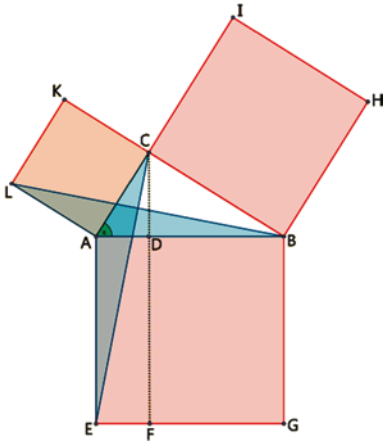

Fig. 4.12 Figure for Pythagoras' theorem, Euclid I,47

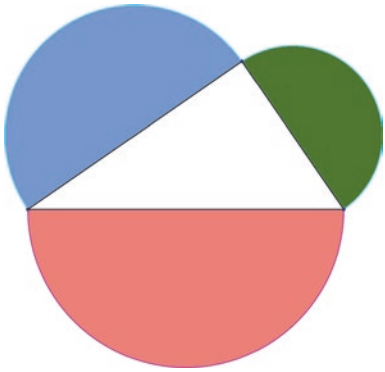

Fig. 4.13 Pythagoras theorem with similar figures

[Euclid VI, 31]. Figure 4.13 shows semicircles as similar figures over catheti or hypotenuse.

Completion by Heron

Heron studied Euclid's proof in detail and additionally proved that in the Pythagoras figure the three lines *AH*, *BL* and *CF* intersect in one point (see Fig. 4.12).

Many proofs of the Pythagorean theorem, based on decompositions into pairs of congruent figures *(cut & paste-geometry)* are particularly intuitive (Fig. 4.14).

The figure in Fig. 4.15, which is probably designed by the famous artist Leonardo da Vinci, is also known. Here the quadrilateral *AIKC* and *ABGD* are congruent; thus the area equality of the hexagons *AIKHBC* resp. *ABGFED* follows by point symmetry resp. reflection. Subtraction of the congruent triangles *ABC*, *IKH* from the first hexagon resp. subtraction of the congruent triangles *ABC*, *ECF* from the second one yields the claim. This corresponds to a rotation about point A.

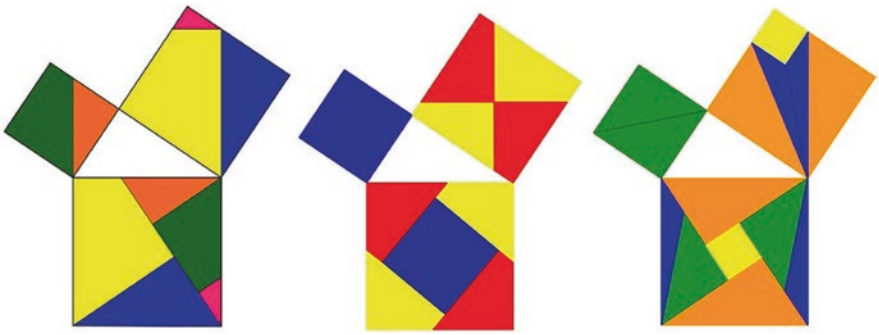

Fig. 4.14 Proof of the Pythagorean theorem by decomposition

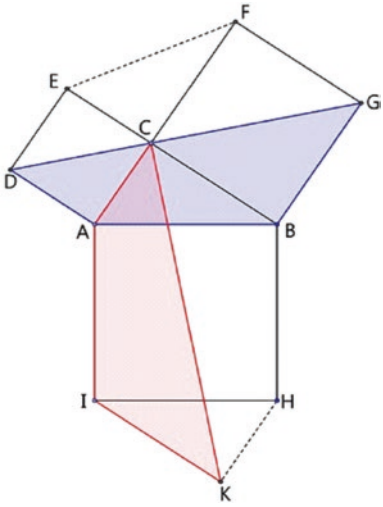

Fig. 4.15 Proof figure after Leonardo

4.6 Pythagorean Number Triplets

Pythagorean triplets are natural numbers $x, y, z \in \mathbb{N}$, which fulfill the following diophantine equation

$$x^2 + y^2 = z^2$$

A possible parameter representation, which Proclus attributes to the Pythagoreans, is

$$x = \frac{1}{2}\left(m^2 - 1\right); \quad y = m; \quad z = \frac{1}{2}\left(m^2 + 1\right)$$

The starting point is the binomial formula:

$$(n+1)^2 = n^2 + (2n+1)$$

Inserting $(2n + 1) = m^2$ and solving for n or (n + 1) shows

$$n = \frac{m^2 - 1}{2} \therefore n + 1 = \frac{m^2 + 1}{2}$$

This gives a Pythagorean triple of the form

$$\left(\frac{m^2+1}{2}\right)^2 = m^2 + \left(\frac{m^2-1}{2}\right)^2$$

Since every value of m generates a triple, this provides a *generator.* A multiple of this is the generator that Proklos attributes to Plato.

$$x = m^2 - 1; \quad y = 2m; \quad z = m^2 + 1$$

Let (a, b, c) be a primitive (coprime) Pythagorean triple; here a, c should be odd and b even. Then it holds

$$a^2 + b^2 = c^2 \Rightarrow b^2 = c^2 - a^2 = (c + a)(c - a)$$

Since b^2 has the divisor 4, it also holds in integer numbers

$$\left(\frac{b}{2}\right)^2 = \frac{c + a}{2}\frac{c - a}{2}$$

Since a, c are coprime, this holds also for $(c - a), (c + a)$. Since a square is on the left, the right side is also the product of two squares. Therefore, we make the ansatz

$$m^2 = \frac{c + a}{2} \therefore n^2 = \frac{c - a}{2}$$

Solving gives the representation for integer $m > n$

$$a = 2mn; \quad b = m^2 - n^2; \quad c = m^2 + n^2$$

This is a 2-parameter generator, as used by Euclid [VIII, 20].

$$x = mn; \quad y = \frac{1}{2}(m^2 - n^2); \quad z = \frac{1}{2}(m^2 + n^2)$$

All Pythagorean triples (a, b, c) (without multiples) can be generated using this generator if (m, n) are relatively prime and of different parity. Figure 4.16 shows that there are 16 relatively prime solutions up to 100.

Fig. 4.16 List of Pythagorean triples

	A	B	C
1	Coprime triplets up to 100		
2	3	4	5
3	5	12	13
4	15	8	17
5	7	24	25
6	21	20	29
7	35	12	37
8	9	40	41
9	45	28	53
10	63	16	65
11	11	60	61
12	33	56	65
13	55	48	73
14	77	36	85
15	13	84	85
16	39	80	89
17	65	72	97

Examples The first triple $(3, 4, 5)$ results from the parameters $m = 3, n = 1$ with $gcd(3, 1) = 1$. This can be seen in

$$x = 3 \cdot 1; \quad y = \frac{1}{2}(9 - 1) = 4; \quad z = \frac{1}{2}(9 + 1) = 5$$

Any multiples (ax, ay, az) generate similar triangles and thus also right-angled. Multiples of the first triple are, inter alia,

$$(6, 8, 10); (9, 12, 15); (12, 16, 20)$$

The area of a right-angled triangle from the above-mentioned triplets is always an integer, since exactly one of the catheti is even and thus the area formula $A = \frac{1}{2}xy$ delivers an integer result. The same also applies to the incircle radius $r = \frac{1}{2}(x + y - z)$, since here exactly two sides are odd and thus the difference is even. The right-angled triangle with the sides (15; 8; 17) has the area $A = \frac{1}{2}15 \cdot 8 = 60$ and the incircle radius $r = \frac{1}{2}(15 + 8 - 17) = 3$.

Search for triplets with given side (cathetus)
Often a Pythagoras triple is sought to a given value x of a cathetus. Because of

$$x^2 + y^2 = z^2 \Rightarrow x^2 = z^2 - y^2 = \underbrace{(z - y)}_{m}(z + y),$$

$m = z - y$ must be a divisor of x^2. The term $x^2 - m^2$ thus results in

$$x^2 - m^2 = \left(z^2 - y^2\right) - (z - y)^2 = 2yz - 2y^2 = 2y(z - y) = 2ym$$

Thus, $2ym$ must also be a divisor of $x^2 - m^2$. If this is the case, then the term $y = \frac{x^2 - m^2}{2m}$ is an integer. The value of z then results from $m = z - y \Rightarrow z = m + y$.

As an **example**, a Pythagorean triple is sought with $x = 15$. If you divide the term $\left(15^2 - m^2\right)$ by all possible values of $2m$ for $m \in \{1, 2, 3, .., x - 2\}$, then $y = \frac{225 - m^2}{2m}$ is an integer for $m \in \{1, 3, 5, 9\}$. From this we obtain by successive insertion the following triple

$$m = 1 \Rightarrow y = \frac{225 - 1}{2} = 112 \Rightarrow z = 1 + 112 = 113 \Rightarrow (x, y, z) = (15, 112, 113)$$

$$m = 3 \Rightarrow y = \frac{225 - 9}{6} = 36 \Rightarrow z = 3 + 36 = 39 \Rightarrow (x, y, z) = (15, 36, 39)$$

$$m = 5 \Rightarrow y = \frac{225 - 25}{10} = 20 \Rightarrow z = 5 + 20 = 25 \Rightarrow (x, y, z) = (15, 20, 25)$$

$$m = 9 \Rightarrow y = \frac{225 - 81}{18} = 8 \Rightarrow z = 9 + 8 = 17 \Rightarrow (x, y, z) = (15, 8, 17)$$

Here we find four solutions. To each integer cathetus $x \geq 3$ at least one corresponding Pythagorean triple can be determined by this algorithm.

4.7 Heronian Triangles and Applications

In calculations of general triangles, it is often disturbing that the altitude (to the base line) is not an integer.

The use of a Heronian triangle composed of two right-angled triangles provides a remedy here. Look for two right-angled triangles with the same leg (cathetus) and assemble them at this leg (as the new altitude) to form a triangle (Fig. 4.17).

The triplets (6; 8; 10) and (15; 8; 17) are used as an example. Assembled at the common leg 8, a Heronian triangle with the sides $a = 10, b = 15, c = 23.$ is obtained. With the even altitude $h_c = 8$, the area A of the Heronian triangle is also an integer.

$$A = \frac{1}{2} g h_c = \frac{21 \cdot 8}{2} = 84$$

This is also confirmed by Heron's formula for the area: With the semi-perimeter $s = \frac{1}{2}(10 + 17 + 21) = 24$ it follows as above

$$A = \sqrt{24(24-10)(24-17)(24-21)} = \sqrt{24 \cdot 14 \cdot 7 \cdot 3} = 84$$

The circumscribed or inscribed radii of the triangle are not necessarily integers, as division occurs here.

$$R = \frac{abc}{4A} = \frac{10 \cdot 17 \cdot 21}{4 \cdot 84} = \frac{85}{8} \therefore r = \frac{A}{s} = \frac{84}{24} = \frac{7}{2}$$

By a suitable composition of Pythagorean triples, one can also construct a cube with integer surface diagonals. Figure 4.18 shows a numerical example attributed to L. Euler.

From Heronian triangles, Brahmagupta was even able to construct Heronian quadrilaterals (Fig. 4.19). He used the triangles (3; 4; 5) and (5; 12; 13), where the hypotenuse of the first triangle is equal to a cathetus of the second. Because of $3 \times 5 = 15$, the first triangle must be enlarged to (15; 20; 25), accordingly the second to (15; 36; 39). This way, the partial triangle BCD is determined. Because of $4 \times 12 = 48$, the first triangle must be enlarged to (36; 48; 60), accordingly the second to (20; 48; 52). Putting them together results in the partial triangle ABD.

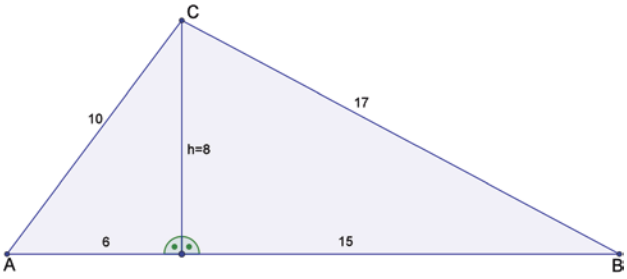

Fig. 4.17 Heronian triangle

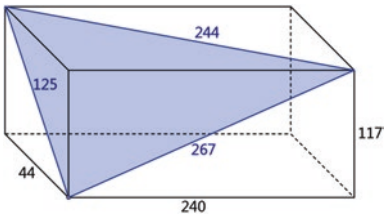

Fig. 4.18 Cuboid with integer surface diagonals

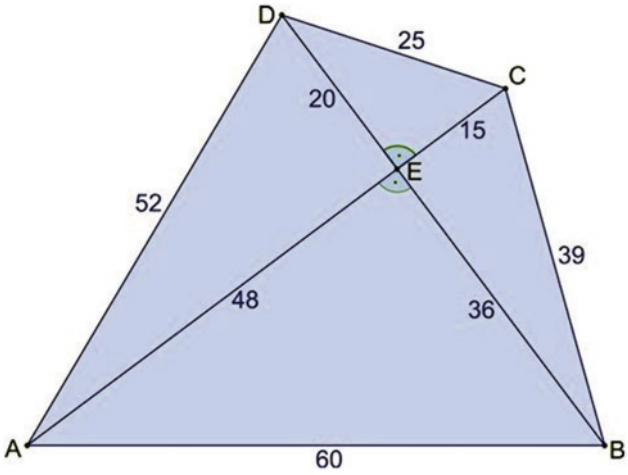

Fig. 4.19 Brahmagupta Quadrilateral

This way, we obtain a quadrilateral with the sides (52; 60; 30; 25) and integer, right-angled intersecting diagonals. If we check the product of the diagonal sections ($48 \times 15 = 20 \times 36$), we find the chord theorem fulfilled. The quadrilateral therefore has a circumcircle!

4.8 Pythagoras and Music

Pythagoras is credited with the discovery that all musical intervals can be expressed in simple ratios of (pure) numbers . Figure 4.20 shows Pythagoras (and Philolaos) playing various musical instruments such as a glockenspiel (chimes), a helikon, and an aulis (flute). Pythagoras adopts this scheme and explains the arithmetic mean of the octave leap $\frac{1+2}{2} = 3{:}2$ as a fifth *(dioxeian)* and the harmonic mean $\frac{2 \cdot 2 \cdot 1}{2+1} = 4{:}3$ as a fourth *(syllaba)*. The fifth and fourth thus add up to exactly one octave

$$(3{:}2) \cdot (4{:}3) = 2{:}1$$

Fig. 4.20 Woodcut by Gaffurio, Theorica musice (1492, Wikimedia Commons)

Together with the root tone, these four tones form the original, classical tetrachord of the Pythagoreans.

By twelve quint steps one reaches the seventh octave; this has the frequency ratio $(2{:}1)^7$, the twelvefold fifth $(3{:}2)^{12}$. However, this does not lead to exactly the same frequency; the relative error is called the *Pythagorean comma*

$$\frac{(3{:}2)^{12}}{(2{:}1)^7} = 1.0136$$

For a quint one has to carry out the quarter and a whole tone. From this one gets the ratio of the whole tone to

$$(4{:}3) \cdot x = 3{:}2 \Rightarrow x = \frac{3}{2}{:}\frac{4}{3} = \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}$$

The octave can then be divided into two quarters and a whole tone (according to Plato, Timaeus 36A,B)

$$(4{:}3)^2 \cdot (9{:}8) = \frac{16}{9}\frac{9}{8} = 2{:}1$$

If one wants to build the octave from six whole tones, one discovers that an octave is something less than six whole tones

$$\frac{2{:}1}{(9{:}8)^6} = 2{:}\frac{531\,441}{262\,144} = \frac{1}{1.0136}$$

This is the reciprocal of the Pythagorean comma! Another subdivision offers the great third with the ratio (5:4). This corresponds approximately, but not exactly, to two whole tones *(= ditonos).* The relative error is here

$$\frac{(9:8)^2}{5:4} = \frac{81}{64} \cdot \frac{4}{5} = \frac{81}{80} = 1.0125$$

This relative error is called the *syntonic comma.* With the advent of polyphony, the *ditonus* was replaced by the more pleasant-sounding major third; this later leads in harmony theory, together with the fifth, to a major chord.

The intervals mentioned above were known by other names to the Pythagoreans and are probably due to Philolaus, as Boethius writes in his treatise on music. But Plato went even further; in determining the semitone, he found the error $\frac{256}{243}$, which he calls λειμμα (*leimma,* remainder). He sets the semitone as the difference between the fourth and the *ditonus*, and obtains

$$\frac{4:3}{(9:8)^2} = \frac{4}{5} \cdot \frac{64}{81} = \frac{256}{243}$$

If the above division is carried out for the fifth, the ratios follow

$$\frac{\frac{3}{2}+1}{2} = 5:4 \therefore \frac{2 \cdot \frac{3}{2} \cdot 1}{\frac{3}{2}+1} = 6:5$$

In musical terms, the fifth is divided into the major and minor thirds. If the *ditonus* is replaced by the minor third, together with the fifth, a minor chord is obtained.

The arithmetic and harmonic division of the major third yields the ratios

$$\frac{\frac{5}{4}+1}{2} = 9:8 \therefore \frac{2 \cdot \frac{5}{4} \cdot 1}{\frac{5}{4}+1} = 10:9$$

The major third is divided into a small and a large whole tone here. The missing subdivisions of the octave are obtained as follows: The small sixth completes the major third to the octave; this shows

$$\frac{2:1}{5:4} = 8:5$$

Analogously, the following applies for the completion to the octave each: The major sixth completes the minor third, so 6:5 follows; the major seventh the major semitone, this shows 16:15; the minor seventh the major whole tone, so 9:8. The large semitone required above is the difference between the major third and the fourth, this provides

$$\frac{4:3}{5:4} = 16:15$$

The *tritonus* stands out from the above schema, the excessive fourth with the ratio 45:32.

In the modern sense, these proportions of numbers can be understood as frequencies, which contradicts the Pythagorean concept of numbers.

The dilemma of the syntonic comma was solved around 1700 by the introduction of equal temperament. In Europe, this was first described by Simon Stevin in 1585; it was introduced in Germany around 1700 by the work of Andreas

Werckmeister for J.S. Bach. Werckmeister emphatically recommends equal temperament, as it

> can be an example of how all devout and well-tempered people live and rejoice with God in always constant and eternal harmony.

The frequencies of the twelve tones of an octave are determined by means of a geometric sequence; each of two adjacent tones receives the frequency ratio $\sqrt[12]{2} = 1.05946$. For the fifth and fourth, there is a small deviation from Pythagorean tuning in equal temperament

$$\left(\sqrt[12]{2}\right)^7 = 1.498301 \approx \frac{3}{2} \therefore \left(\sqrt[12]{2}\right)^5 = 1.33484 \approx \frac{4}{3}$$

The table of all tones follows:

Interval	Tone	Pythagorean	Equal temperament
Fundamental tone	C	1	$\left(\sqrt[12]{2}\right)^0 = 1$
Semitone (small second)	$C^{\#}$	$\frac{16}{15} = 1.06667$	$\left(\sqrt[12]{2}\right)^1 = 1.05946$
Tone (large second)	D	$\frac{9}{8} = 1.125$	$\left(\sqrt[12]{2}\right)^2 = 1.12246$
Minor third	$D^{\#}$	$\frac{6}{5} = 1.2$	$\left(\sqrt[12]{2}\right)^3 = 1.18921$
Major third	E	$\frac{5}{4} = 1.25$	$\left(\sqrt[12]{2}\right)^4 = 1.25992$
Forth	F	$\frac{4}{3} = 1.33333$	$\left(\sqrt[12]{2}\right)^5 = 1.33484$
Tritonus	$F^{\#}$	$\frac{45}{32} = 1.40625$	$\left(\sqrt[12]{2}\right)^6 = 1.41421$
Fifth	G	$\frac{3}{2} = 1.5$	$\left(\sqrt[12]{2}\right)^7 = 1.49831$
Minor sixth	$G^{\#}$	$\frac{8}{5} = 1.6$	$\left(\sqrt[12]{2}\right)^8 = 1.58740$
Major sixth	A	$\frac{5}{3} = 1.66667$	$\left(\sqrt[12]{2}\right)^9 = 1.68179$
Minor seventh	B	$\frac{9}{5} = 1.8$	$\left(\sqrt[12]{2}\right)^{10} = 1{,}78180$
Major seventh	H	$\frac{15}{8} = 1.875$	$\left(\sqrt[12]{2}\right)^{11} = 1.88775$
Octave	C'	2	$\left(\sqrt[12]{2}\right)^{12} = 2$

The view of the cosmos inspired the Pythagoreans to assign certain tone intervals to the planetary orbits, which together form a heavenly harmony, later called the music of the spheres by Eudoxus. It remained unclear whether the ratio of the planetary orbits or their orbital periods was determinative for the harmony. Plato adopts the Pythagorean ideas in his works *Politeia* [616b] and *Timaios* [35a]. Although Aristotle argues against the music of the spheres being in the audible range (*De Caelo* 290b) *Sphärenmusik*, the theory found numerous followers first in the Greek and Roman world. In modern times, the topic occupied not only Kepler but also famous poets such as Dante, Shakespeare and Goethe. Especially Cicero discussed this question in detail in his writings *De re publica* (6.17) and *De natura Deorum* (2.7.19 ff).

The late-Pythagorean *Nikomachos* further developed Pythagorean music theory in his *(Ἐνχειρίδιον ἁρμονικῆς)* and introduces no less than 28 tone names. In Chap. 6 of the *Harmonics*, Nikomachos tells the well-known legend that Pythagoras listened to the tones generated by the hammers while passing by a forge. He noted the weights of the hammers whose produced sound had a harmonic effect. According to Flora R. Levin[11], the contribution that Nikomachos made to Pythagorean music theory is underestimated. In particular, she is of the opinion that the mathematical foundations for octave, fifth and fourth ultimately go back to Nikomachos. This greatly reduces Plato's influence.

Nikomachos' harmony was probably the most important source for the treatise *De Institutione musica* by Boethius. At the beginning of the work are poetic lines:

> And first of all, the music of the world must be examined here, which is experienced in heaven itself and also in the structure of the elements and in the change of times:
>
> How can it be, after all, that the rapid heavenly machinery runs so quietly and silently?
> And if that sound does not reach our ears, which would be necessary for many reasons,
> how could such a rapid movement of such large bodies not cause a single tone?

Pliny the Elder (23–79) had previously assigned the intervals of the planets to Pythagoras (*Natural History,* II, 3, 6):

> But Pythagoras sometimes determined the distances according to musical laws and called the distance from the earth to the moon a whole tone, from the moon to Mars a half tone, from Mars to Venus a half tone, from Venus to the sun three half tones, from the sun to Mars a whole tone, from Mars to Jupiter a half tone, from Jupiter to Saturn a half tone, and from Saturn to the zodiac three half tones. In this way, seven tones are produced, which are called the complete harmony.

[11] Levin F.R.: *The Harmonics of Nicomachus and the Pythagorean Tradition,* University Park 1975, pp. 46–50.

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Hippocrates of Chios

5

Hippocrates of Chios – not to be confused with his namesake, the physician Hippocrates of Kos – lived in the second half of the fifth century BC. He was a somewhat naive businessman who lost his fortune and therefore moved to Athens for studies. Aristotle reports in [Ethics ad Eudem., VII, c.14]

> So Hippocrates was a good geometer, while he seems to be simple-minded and ignorant in the rest; at least, it is said, through credulity he lost a large sum of money to the customs officers of Byzantium.

J. Philoponus, on the other hand, writes in *(Comm. In Arist. physicae auscultatio)*

> Hippocrates of Chios, a merchant, fell into the hands of a pirate ship, lost all his possessions and went to Athens to sue the robbers; since he now had to stay in Athens for the sake of the lawsuit and often visited the schools of philosophers, he gradually came to such a high level of geometric knowledge that he tried to find the quadrature of the circle.

Three mathematical achievements of Hippocrates have remained known:

- Doubling of a cube by inserting two middle proportionals (which is also ascribed to others)
- Attempt at the quadrature of the circle using Möndchen
- Composition of the *Elements* of Mathematics

The works of Hippocrates are not preserved. The only clues were provided by the lecturer of the new academy *Simplikios* (580 to 640 AD), from the province of Cilicia, who wrote numerous valuable comments on Aristotle. In one of these comments[1] on Aristotle's *physica auscultatio* he added a report on the

[1] Simplikios, H. Diels H. (ed.): In Aristotle's Physicarum laboris quattuor priores commentaria, Berlin 1932.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_5

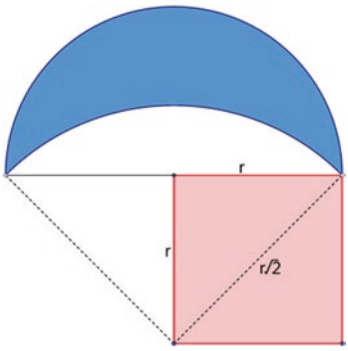

Fig. 5.1 Hippocrates' Construction 1

quadrature attempts of Antiphon and Hippocrates, which draws on two sources. One is a report by the Aristotle commentator *Alexander* of Aphrodisias (second century AD), the other the Catalogue of geometers of Eudemos (around 320 BC). Alexander was appointed head of the Lyceum during the reign of Emperor Septimius Severus (198–209 AD) and therefore dedicated his work *About destiny* to the emperor.

In his commentary, Simplicius assures to quote Eudemus almost verbatim, with occasional additions. This commentary was unknown until 1860 and was therefore not included in the 1802 edition of *Histoires des Mathématique* by J.-E. Montucla. The German mathematician C.A. Bretschneider[2] found the only surviving copy of Simplicius's commentary and published it in 1870. The included passages from Eudemus are among the oldest pieces of mathematical literature that have been handed down and are therefore of particular interest. A presentation can be found in W.R. Knorr[3] and I. Thomas[4], who later called himself Bulmer-Thomas.

In reference to proposition [Euclid I, 45], Proclus writes:

> After receiving knowledge of this problem [I, 45], I believe that the ancients attempted to square the circle. For, if parallelograms are equal in area to any rectangles, it is worth investigating whether one can prove that rectangles are equal in area to figures that are bounded by arcs of a circle.

[2] Bretschneider C.A.: Die Geometrie und die Geometer vor Euklides, Reprint Sändig 2002, S. 100–127.

[3] Knorr W.R.: The Ancient Tradition of Geometric Problems, Dover 1993, S. 30–41.

[4] Thomas I.: Selections illustrating the History of Greek Mathematics I, William Heinemann 1962, S. 234–251.

5.1 Quadratures According to Alexander of Aphrodisias

Construction 1
Upon the hypotenuse of a right-angled isosceles triangle, a semicircle is drawn; the apex of the triangle is the center of a quarter circle. The resulting little cup is equal in area to the triangle or to the square above the half hypotenuse (Fig. 5.1).

Proof It holds that the area is equal:

$$lunule + quarter\,circle = semicircle + triangle$$
$$\Rightarrow lunule = semicircle + triangle - quarter\,circle$$

With the radius r of the semicircle, it follows in modern notation

$$lunule = \frac{1}{2}\pi r^2 + r^2 - \frac{1}{4}\pi\left(r\sqrt{2}\right)^2 = r^2$$

The construction is also often represented with a triangle that is mirrored upwards (Fig. 5.2):

Since, according to Hippocrates, similar arc segments behave like the squares of the radii, the following applies to the area ratio

$$\frac{segment(AC)}{segment(AB)} = \left(\frac{r\sqrt{2}}{2r}\right)^2 = \frac{1}{2}$$

The arc segments are called similar if the inscribed triangles are similar. The larger arc segment over AB is therefore equal in area to the sum of the two smaller ones. If you add the crescent-shaped area to the arc segment AB, you get a semicircle. You also get a semicircle if you add the right-angled isosceles triangle to the smaller arc segments. This shows once again that the areas of the triangle and the crescent are equal.

Construction 2
The following construction is remarkable. An isosceles trapezoid is given as half of a regular hexagon. Circles are constructed over the three (congruent) sides of the trapezoid, which together with the circumference of the trapezoid form three

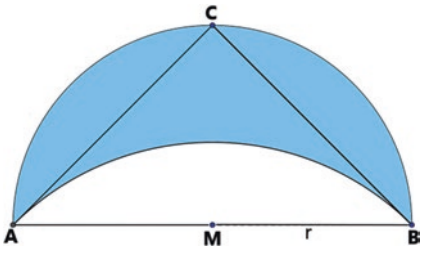

Fig. 5.2 Hippocrates' 1 (modified)

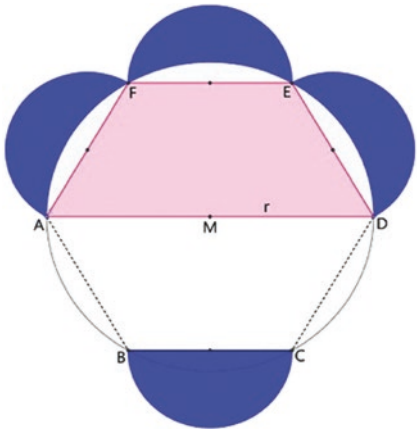

Fig. 5.3 Hippocrates' Construction 2

crescents. The sum of the areas of these three crescents together with a semicircle over the circumference radius is equal in area to the trapezoid! (Fig. 5.3)

Proof The area of a crescent is obtained here from

$$\begin{aligned} &\textit{semicircle} - \textit{lunule} = \textit{one sixth circle} - \textit{isosceles } \triangle \\ &\Rightarrow \textit{lunule} = \textit{semicircle} + \textit{isosceles } \triangle - \textit{one sixth circle} \end{aligned}$$

$$\begin{aligned} &\textit{lunule} = \tfrac{1}{2}\pi\left(\tfrac{r}{2}\right)^2 + \tfrac{1}{4}r^2\sqrt{3} - \tfrac{1}{6}\pi r^2 \\ &\Rightarrow \textit{lunule} = \tfrac{1}{4}r^2\sqrt{3} - \tfrac{1}{24}\pi r^2 \end{aligned}$$

The sum of the areas S of the 3 crescents increased by the semicircle over $|BC|$ is therefore

$$S = \frac{3}{4}r^2\sqrt{3}$$

The area of the trapezoid is equal to the sum of the areas of three congruent triangles

$$\textit{Trapez} = \frac{3}{4}r^2\sqrt{3}$$

This proves the desired area equality.

5.2 Quadratures According to Eudemos

Construction 3
The following construction is also ingenious. In a trapezoid with 3 congruent sides $|BC| = |CD| = |AD|$ is given

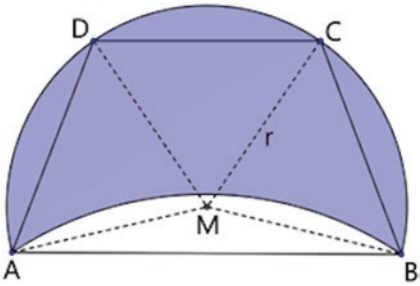

Fig. 5.4 Hippocrates' Construction 3

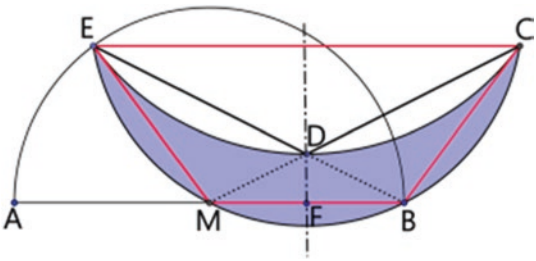

Fig. 5.5 Hippocrates' Construction 4

$$|AB|^2 = 3|AD|^2 \Rightarrow |AB| = \sqrt{3}|AD|$$

The circle arc over AB is similar to the circle arc AD; this means that the segment AB is equal in area to three times the segment AD. The isosceles trapezoid $ABCD$ is a lune, since the adjacent angles at parallels to 180° supplement each other; therefore, the trapezoid has a circumcircle.

Claim: The crescent, formed from the circumcircle of the trapezoid and the circle arc over the base AB, is equal in area to the trapezoid (Fig. 5.4).

According to the specification: For the semicircle applies: semicircle(AB) = trapezoid + 3 circle segment(AD) - circle segment(AB) = trapezoid

Construction 4

A *Neusis* construction is the following: Given is a semicircle with radius AM. The point D is initially located on the symmetry axis to MB; likewise, E is on the semicircle. With the help of a ruler, the distance ED is now inserted so that B lies on the extension and it holds $|ED|^2 = \frac{3}{2}|AB|^2$. The point C is the point of reflection of E with respect to the median of MB. The outer circle arc through M, B via EC is the circumcircle of the trapezoid $MBCE$. The inner circle arc through D via EC is determined by the fact that the circle section through M, B is similar to the one through D.

Claim: The crescent is congruent to the pentagon $MBCDE$ (Fig. 5.5).

The proof is left to the reader. Among other things, it is to be shown that the triangles ΔDFB or ΔAEB are similar and the points A, F, D, E are concyclic (see T. Heath I, p. 193-195).

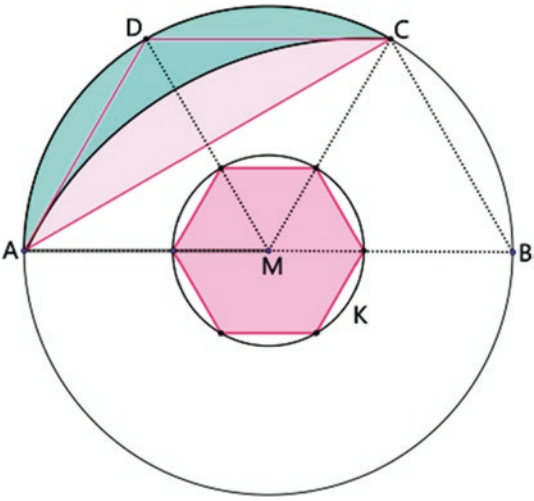

Fig. 5.6 Hippocrates' Construction 5

Construction 5

The following construction is also surprising:

Given are two concentric circles, where the outer radius is $\sqrt{6}$ times the inner. A regular hexagon is inscribed in the inner circle K. Three adjacent points of the hexagon inscribed in the outer one are to be A, C and D. The circle section erected over the chord AC is similar to the circle section over AD again.

Claim: The sum of the areas of triangle ACD and the inner hexagon is equal to the lune plus the small circle K (Fig. 5.6), see (Ivor Thomas, Selections illustrating the History of Greek Mathematics, p. 249-253).

According to Simplicius, Hippocrates thought that he had accomplished the squaring of the circle. With the known lune area, the area of the small circle could be constructed exactly given the triangle and hexagon. However, when considering the sophisticated constructions that Hippocrates invented, most authors believe that Simplicius was wrong here. Hippocrates certainly recognized that with this type of task, the general problem of the squaring of the circle *has not* been solved. Aristotle describes the squaring of the circle by means of a lune as a fallacy [On the soph. Refutation 171b12–16].

In his proofs, Hippocrates shows knowledge of the following theorems:

1. To similar segments belong congruent angles; their areas are related as the squares of their tendons.
2. Angles at circumference in semicircles are right angles; to segments smaller than a semicircle belong acute angles
3. The side of a hexagon inscribed in a circle is equal to the radius of the circle
4. In a triangle with an acute angle, the square on the opposite side is smaller than the sum of the squares of the adjacent sides

5. In a triangle with an obtuse angle, the square on the opposite side is greater than the sum of the squares of the adjacent sides
6. In an equilateral triangle, the square on the altitude is three times the square on the half-base
7. Circular areas behave like the squares of their diameters
8. In similar triangles, corresponding sides are in the same ratio
9. The isosceles trapezoid can be inscribed in a circle

These sentences were certainly contained in the (lost) *elements* of Hippocrates.

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Athens and the Academy

6

O shining, violet-crowned, sung-about city,
Fortress of Hellas, glorious Athena
over the city her moon stands, pouring out its light
on the theater's stone steps on the south slope of the Acropolis.
The chorus moving in dance,
with torches under the night sky.
And its light shines further
beyond the cypresses, by the columns of Zeus
and beyond the gate of the Greek-loving Hadrian.
(Ode of the poet Pindar to the city of Athens)

6.1 Athens

The origin of the name *Athens* (ancient Greek Αθήναι, today Αθήνα) has not yet been clarified, as the etymological roots of the term are unclear. It is assumed that the name derives from the city's protective goddess, Athena, but it is also possible that the goddess was named after the city. In ancient times, the plural Athinai was used, also for residents of the Attic peninsula. Since about 1300 BC, there was a Mycenaean palace on the Acropolis hill. The oldest city layout was limited to the upper area of the rock hill, which later became the military and religious center of Athens as the Acropolis (ἀκρῐ́ς = hilltop, πόλις = city). Figure 6.1 shows Athens and the Acropolis after the reconstruction by Augustus and Hadrian.

The Athenians praised the achievement of the (legendary) Theseus, the "synoecism", that is, the "settling together" of the population of Attica, which until then had been scattered over a number of villages with their own constitution, as an outstanding achievement. With this goal, Theseus abolished the individual administrations of the villages and concentrated all power in the common authorities in Cecrops, which now was called Athens. This event has been celebrated since then in the Pan-Athenian Games and in the *synecia* for the goddess Athena, which Erechtheus has established; he also founded the Erechtheion on the Acropolis.

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Fig. 6.1 Athens and Acropolis around 100 AD (Wikimedia Commons)

A rule of the nobility with a king at its head developed. The nobility, represented by an aristocratic council (*Areopagus*), regularly gathered on the Ares Hill to influence state affairs. The Areopagus determined the first officials, called archons, and the military commander.

Due to its central location and its active participation in maritime trade via the port of Piraeus, Athens soon became prosperous, though not all Athenian citizens benefited from this. This led to social tensions between landowners and the landless laborers who had no rights. Under the rule of Draco, the need for reform was recognized; his successor Solon achieved a balance of interests between the nobility and the citizens of the city with his draft constitution. The Athenians were divided into four classes, of which only the upper class was admitted to political office, at the same time slavery on the plantations was abolished. In the assembly of the people, all classes had the right to vote, thus creating beginnings of democracy in the Athenian constitution. Although Peisistratus managed to seize power again as a tyrant in 541 BC, the tyranny ended with the death of his two sons.

Persia, Greece's eastern neighbor, began its expansion under Cyrus II. Under his leadership, the Median Empire was conquered in 530 BC, the Lydian Empire under King Croesus in 546, and Babylon in 539. Cyrus' successor Cambyses II also conquered Egypt in 525. With the conquest of the Lydian Empire, all of Asia Minor was in Persian hands, so all Ionian cities had to pay tribute. Athens supported the Ionian uprising led by Miletus, but the Persians completely destroyed Miletus under Darius I in 494 BC and sold the inhabitants into slavery.

The most well-known battles between Athens and Persians are those of Marathon (490 BC) and Salamis (480 BC). After the battle of Salamis, the Athenians were able to expand their power to other cities. Athens became leading in the Attic Sea Alliance, founded in 477 and rebuilt in 448. The peace treaty of 448 with Persia (so-called *Kallias Peace*) and the peace treaty of 445 forced by Sparta (for 30 years) gave Athens a break for recovery, which is called the Periclean Age from 443 to 429 BC. In this time *Perikles* started with the construction or expansion of the Acropolis, which had been previously destroyed by the Persians. The construction was led by the architects Kallikratos and Iktinos, and the equipment was provided by the famous sculptor Phidias (construction time 467–406). Figure 6.2 shows the interior of the Parthenon with the gilded giant statue of the goddess *Athena* Parthenos (= born from the head of Zeus).

Fig. 6.2 Statue of Athena in the Parthenon (Wikimedia Commons)

Athens reached the height of its political and cultural influence in the fifth century BC. It became a center for scientists, writers, philosophers, and artists. Famous dramatists were Aeschylus (525–456), Sophocles (497–406), Euripides (480–406), and Aristophanes (445–385). The history was founded by Herodotus (484–425), Thucydides (460–396), and Xenophon (430–354). The philosophy made Socrates (469–399) popular; he was followed by Plato and Aristotle. Both founded a significant Academy, the name Academy is used for scientific institutes later for centuries. Athens had a population of approximately 40,000 inhabitants, but most Athenians were not full citizens, but rather slaves or foreigners (metoecs).

After the expiration of the peace treaty, there is again a confrontation with the arch-enemy Sparta in the Peloponnesian War (431–404), which ends with a heavy defeat and the occupation of Athens. Despite the difficult times, Plato is able to founded the Academy in Athens in 387 (see city map Fig. 6.3), which becomes a magnet for students from all over Greece. In 338, Philip of Macedon defeats the Hellenic League consisting of Sparta, Athens and Thebes in the Battle of Chaeronea. From 334 onwards, Athens is newly occupied, and Aristotle (without citizenship) is able to found the Lyceum (see city map). Athens is now part of the Macedonian Empire and has to provide considerable troops for Alexander's campaign.

Philipp and after him his son Alexander the Great respected the great tradition of Athens and spared the city. After the death of Alexander, Athens and other Greek city-states rose up again against the Macedonians in the so-called Lamian War, but were defeated. Athens received a Macedonian occupation: From 317 to 307 BC, Demetrius of Phaleron, a student of Aristotle, ruled as Macedonian governor. After the expulsion of Demetrius, Athens became independent again and successfully defended itself against Macedonian conquest attempts until 262 BC.

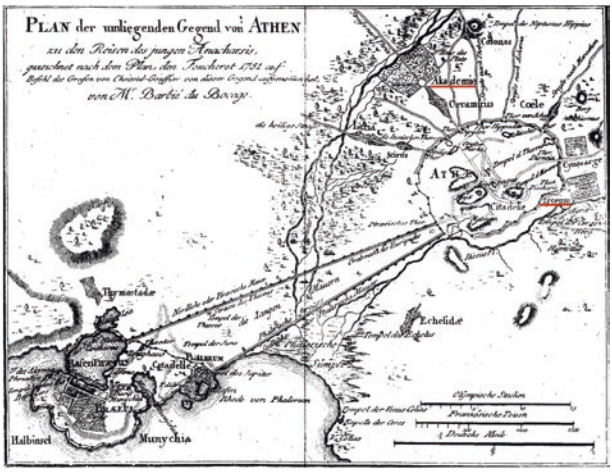

Fig. 6.3 City map of Athens from 1571 (Wikimedia Commons)

After another defeat, Athens was Macedonian again from 229 BC, then it was free. The independent Athens regained cultural importance and became an important political arena for the Diadochi and their successor states, which competed for the favor of the Greek public with numerous foundations. In the conflicts of Macedonia with the Roman Empire, Athens was on the side of Rome.

In 86 BC, Athens was conquered and sacked by Sulla as punishment for supporting Mithridates VI. and was looted. This made Greece a Roman, later an Eastern Roman province (from 395 AD). However, even under Roman occupation, the city remained a center of cultural life and had the status of a free city. Many members of the Roman upper class went to Athens for a while to study philosophy. Numerous new buildings and renovations were carried out in Roman times. The Roman agora (under Augustus) and the new Hadrian city district (under Hadrian) were built. Finally, the temple of Olympian Zeus (*Olympieion*), which had already begun in the Archaic period, was completed.

6.2 The Academy

The Academy was originally designed to be permanent; i.e. beyond the death of Plato. In the 1781 Athens city map (Fig. 6.3), the location of the Academy (northwest of the ancient city walls) is still clearly visible; as is the location of the Lykeion, here referred to as the "Lyceum". The Academy Board was elected from the group of Elders each time. The teaching was mostly public: lectures, dialogues, disputes and speeches alternated. There was no fixed curriculum. Not only philosophy, but almost every scientific field (still counted as philosophy) was taught and researched, such as mathematics and astronomy. The numerous

celebrations and symposia on various occasions – such as the birthdays of Socrates and Plato – were probably not public.

Plato's successor was *Speusippos;* his successors were Xenocrates of Chalcedon (339–315), Polemon of Athens (315–270) and Crates of Athens (from 270). After that, the old Academy died out. It is by no means the case, as is often read in the literature, that the Academy was closed in 529 after continuous existence. Rather, it existed in six different periods:

1. The continuation of the *old* Academy from 347–270; its best representatives are *Xenocrates* and *Krantor.*
2. The *middle* Academy was founded by *Arkesilaos* (316–241). The doctrine deviates from Plato's dogma and turns to skepticism. His successor *Karneades the Elder* (156–137), who was sent as an ambassador to Rome, was so successful there in his teaching that the famous Cato the Elder had him expelled from Rome as a troublesome competitor.
3. The *newer* Academy was led by *Karneades the Younger* (214–129). The doctrine rejects Plato's dogma; the pure truth is in principle inaccessible, it is enough to investigate that, what is probable.
4. The *fourth* academy was founded by *Philon* of Larissa and *Antiochos* of Askalon after the conquest of Athens by Sulla (86 BC). The doctrine was again oriented towards Plato. Both philosophers also traveled to Rome, where they found numerous listeners.
5. The *fifth* academy was founded by *Askalon,* a student of Philon. Its aim was to unite the concerns of the Stoics with the Platonic doctrine.
6. The *neoplatonic* academy was newly founded in 410 AD by the patron *Plutarch* of Athens as a bastion against the emerging Christianity. The successors of Plutarch were *Syrianos* of Alexandria, *Domninos* of Larissa, *Proklos* Diadochos, *Marinos* of Sichem, *Isidore of Miletus* and finally *Damaskios* from Damascus (510–529).

In addition to the Peripatos of Aristotle, there are two other schools writing history of philosophy:

a) The *Stoa* (= column gallery) was founded by Zeno of Kition (332–262 BC); successors were Chrysippus and Poseidonius. The school of the Stoics became popular in Rome, where Seneca, Epictetus and the emperor Marcus Aurelius became followers.
b) The *Kepos* (= *Garden)* of Epicurus (341–270 BC) represents the doctrine that true pleasure consists of "freedom from bodily pain and mental agitation". The philosopher Nietzsche (*Untimely Meditations*, 192) formulated the concern of the Epicureans wittily: *A garden, figs, a small cheese, plus three or four friends – that was the opulence of Epicurus.*

When Cicero was in Athens in 78–77 BC, he attended the lectures of *Antiochus*. In his work *De finibus bonorum* (V, 1,1) Cicero reports how he, with his brother and nephew, visited the old Academy and found the still existing tomb of Plato:

> Once I had been attending a lecture of Antiochus, ...in the building called the School of Ptolemy; and with us were my brother Quintus and my cousin Lucius... We arranged to take our afternoon stroll in the Academy, mainly because the place would be quiet and deserted at that hour of the day... When we reached the walks of the Academy, which are so deservedly famous, we had them entirely to ourselves, as we had hoped...I was reminded of Plato, the first philosopher, so we are told, who held discussions in this place; and indeed the garden close at hand yonder recalls his memory.

At the Neoplatonic Academy, *Proclus* (around 450 AD) worked, whose commentary on Book I of *Elements* became an important source of the history of mathematics. Also known is *Isidore* of Miletus, who was one of the architects in the reconstruction of the Hagia Sophia (after its collapse during the Nika revolt in 532). During the tenure of the last Academy president, the mathematician *Simplicius* also worked at the Academy, to whom we owe an extensive commentary on Aristotle's natural science writings. The Academy was closed by Emperor Justinian in 529 AD. With this, the Hellenistic culture was finally ended in Athens, about 140 years after the closure of the Serapeion in Alexandria.

The Christianity in the Eastern and Western Roman Empire had become state religion; the old scriptures were considered as pagan evidence. Mathematics was equated with astrology; geometry was seen as the *true* mathematics. A. Augustine (of Hippo) writes:

> The good Christian should beware of the mathematicians and all those who make empty predictions, especially if these predictions come true. For there is a danger that the *mathematicians* in league with the devil cloud the mind and draw into the bonds of hell.

Not only mathematics, also the tradition of the spherical shape of the earth was considered dangerous for the salvation of the soul. So the monk *Cosmas* Indicopleustes writes in the sixth century in his work "Christian Topography":

> Whoever wants to be a true Christian must abandon the geometric methods of the fools and liars. To those who want to be Christians and yet despise the word of God, God will say on the day of judgment according to the apostle Matthew: *I do not know you, depart from me, you who practice evil.*

Tertullian (Apologeticum, 40) gives the pagans partial blame for all evil:

> That the general calamities and distress are sent by the gods out of anger against the Christians is mere delusion, as history shows. The real blame lies with the general guilt, especially that of the pagans. The Christians are to thank that it is not worse.

Theoderet of Cyr writes in his work[1] *Remedies against Hellenistic diseases:*

[1] Théoderet de Cyr, Canivet P. (ed.): Thérapeutique des malades Hélleniques, 8,68, Sources Chrétiennes Band 57, Paris 1958.

> Truly, their temples are so completely destroyed that one cannot even imagine their former site, while the building material is now dedicated to the shrines of the martyrs. … See, instead of the feast for Dionysios and others, the public events are now celebrated in honor of Peter, Paul and Thomas! Instead of practicing lewd customs, we now sing chaste hymns of praise.

6.3 The Mathematicians of the Academy

Thanks to the mathematicians catalogue of *Eudemos* (partially transmitted by Proklos) we have a good overview of the mathematicians working at the Academy; three of them are introduced here.

A famous mosaic based on a Greek model from Torre Annunziata near Pompeii (first century AD) is usually seen as a representation of the Academy of Plato. Maybe the Seven Sages of Greece are depicted here (Fig. 6.4).

Three of the scholars are holding papyrus scrolls that they have taken from the open box in the foreground. Also in the foreground is a globe (σφαῖρα) with a coordinate system, to which one of the scholars is pointing with a stick. A column in the background bears a sundial; on the left arch in the picture are several large oil lamps that provide light in the dark.

Fig. 6.4 Mosaic of the "Academy of Plato" (Wikimedia Commons)

The motif of the "Seven Sages" seems to have been very popular, as a similar copy of the mosaic (without a tree in the middle) exists in the Villa Albani (Rome).[2]

6.4 Eudoxus of Knidus

Eudoxus (Εὔδοξος) came from the peninsula of Knidus (Asia Minor) and initially received a medical education. His year of birth can be approximately set at 395 BC, as he explicitly refers to Plato's death in his work. Since Diogenes Laertius (VIII, 86) gives his age as 53 years, he probably died around 340. At the age of about 23, he went to Athens to listen to the Socratics teaching there. This first stay in Athens lasted only a few months, as the meeting with Plato turned out to be disappointing for him. The exact reason for leaving the Academy is not known; anyway, Eudoxus soon returned to Knidus. Supposedly, after a long journey to Egypt together with the physician *Chrysippus* from Knidus, he took up a teaching position in Zyzikus (Marmara coast).

He acquired his knowledge of astronomy in Egypt. In this subject, his theory is significant in which he explained the different angular velocities of the planets by using countless concentric spheres around the earth at rest in the center. His writing about the movement of the planets are lost. It can be partially reconstructed from Aristotle's Metaphysics (XII, 8) and Simplikos' commentary on *De Caelo*.

Little is known about his later stay at the Academy; it is reported that Aristotle entered the Academy during the time of Eudoxus. Aristoxenus of Tarentum reports that Eudoxus taught the subject of mathematics during Plato's second Sicilian expedition. This seems possible in terms of age, but is hardly plausible given the short duration of his stay in Athens. After another period of teaching in Zyzikus, he returned to his homeland where he also became involved in politics.

The mathematical work

Only fragments of his mathematical discoveries are known from the works of other authors. His contributions to mathematics are significant because he provided a comprehensive theory of proportions. It is assumed that Books V and XII of the *Elements* are essentially from him. Furthermore, the following theorems are from him:

a) Circular areas behave like the squares of the radii [Euclid XII, 2].
b) Spherical volumes behave like the cubes of the radii [Euclid XII, 18].
c) The volume of a pyramid is one third of the prism with the same base and height [Euclid XII, 3–7].
d) The volume of a cone is one third of the cylinder with the same base and height [Euclid XII, 10].

[2] Stückelberger A.: Einführung in die antiken Wissenschaften, Wissenschaftliche Buchgesellschaft 1988, Tafel VII.

The fundamental problem that arose in the treatment of proportions before Eudoxus was the linking of all sizes with their respective units. Previously, it was only possible to set equal quantities such as lengths, areas and volumes in proportion to each other. Aristotle writes in [Anal. post. I, 5] about the theorem of proportions:

> Previously, this theorem was proved individually for numbers, lengths, bodies and times. Only after establishing a general concept of size, this theorem could be proved generally.

Similar in Aristotle [Topika 158B]:

> It seems that in mathematics, some things are difficult to prove because of the lack of a [suitable] definition, for example, that a line that intersects a surface [parallelogram] parallel to one side, divides the length and the area in the same proportion. But if the definition is pronounced, what has been said is immediately clear, because the surface areas and length have the same *anthyphairesis*.

Anthyphairesis is called *mutual subtraction* and characterizes exactly the procedure for the Euclidean algorithm. This process terminates exactly in a finite number of steps if the ratio of the quantities is rational. But what if the ratio is irrational? The Greeks had problems with declaring infinite processes in mathematics as valid. Irrational numbers like $\sqrt{2}$ could be expressed by a proportion:

$$x : 2 = 1 : x$$

Rules for proportions

The new, generally valid definition can be found in Book (V,5) of the *Elements*: We say that quantities are in the same proportion if the first is to the second as the third is to the fourth. Expressed in formulas, $a{:}b = c{:}d$, implies that exactly one of the three following cases occurs

$$na > mb \Leftrightarrow nc > md$$

$$na = mb \Leftrightarrow nc = md$$

$$na < mb \Leftrightarrow nc < md$$

Here *n, m* are any integer multiples. In particular, the commutability can be shown

$$a{:}b = c{:}d \Leftrightarrow c{:}d = a{:}b \vee b{:}a = d{:}c$$

Likewise, the corresponding addition or subtraction is to be proved:

$$a{:}b = c{:}d \Leftrightarrow (a + b){:}b = (c + d){:}d$$

$$a{:}b = c{:}d \Leftrightarrow (a - b){:}b = (c - d){:}d;\ a > b; c > d$$

As an application, [Euclid V, 8] is proven: From two unequal quantities, the larger has to a fixed quantity a larger ratio than the smaller; and the fixed quantity has to the smaller a larger ratio than to the larger.

For the proof, we need the Axiom of Archimedes, which probably comes from Eudoxus:

> If two segments are on a line, then the smaller can exceed the larger, if the smaller is often enough applied on the line.

Written with logical quantifiers, it reads as follows:

> For each two similar quantities *a, b* there exists (at least) one natural number *n* so that

$$0 < a < b \Rightarrow \exists\, n \in \mathbb{N} : na > b$$

As an example, it shall be proven: If $a > b > 0, c > 0 \Rightarrow \frac{a}{c} > \frac{b}{c}$

Case 1) $a - b < b$. According to the Archimedes axiom, there is a multiple m with

$$m(a - b) > c \tag{6.1}$$

Likewise, there is a smallest multiple nc that exceeds mb

$$nc > mb \Rightarrow (n - 1)c < mb \tag{6.2}$$

With (6.1) and (6.2) it follows

$$ma - mb > c \Rightarrow ma - (n - 1)c \geq c \Rightarrow ma \geq nc$$

Case 2) $b < a - b$. There is a multiple m with

$$mb > c \tag{6.3}$$

Likewise, there is a smallest multiple nc that exceeds $m(a - b)$

$$nc > m(a - b) \Rightarrow (n - 1)c < m(a - b) \tag{6.4}$$

With (6.3) and (6.4) it follows

$$ma - mb > (n - 1)c \Rightarrow ma - c \geq (n - 1)c \Rightarrow ma \geq nc$$

In both cases, therefore, $ma \geq nc$. Also, $mb \leq nc$, is true because in the second case it follows that $m(a - b) > mb$. Therefore, in total

$$ma > nc \;\therefore\; mb > nc \Rightarrow \frac{a}{c} > \frac{b}{c}$$

Eudoxus is the author of the principle of the exhaustion proof (cf. Archimedes' measurement of a circle). In the formulation of [Euclid X, 1] it says:

> If, when two unequal quantities (of the same kind) are given, one takes away from the greater a part which is greater than half, and from the remainder another part which is greater than half of that, and so on continually, there must at last remain some quantity which is less than the less quantity originally given.

6.5 Theodorus of Cyrene

According to the report of Iamblichus, Theodorus (465–399 BC) was a Pythagorean and one of Plato's teachers of mathematics. In the dialogue *Theaetetus* [147E] the latter reports:

> Theodorus drew something for us about the diagonals of squares, by showing us, with regard to that with an area of three square feet and that with an area of five square feet, that they are not commensurable in length with that with an area of one square foot. And so he went through each one in turn up to the [diagonal] with an area of seventeen square feet, at which he stopped. Now, since it seemed that there were an infinite number of lengths, we wanted to try to find a general concept by which we could designate all of them.

Most literature interprets this passage that Theodorus constructed squares with areas 3 to 17, and demonstrated the irrationality of the diagonal and side lengths for these areas (excluding squares 4, 9, and 16). Furthermore, it is assumed that Theodorus showed the incommensurability in individual cases, while Theaetetus treated the general case:

> Lines over which erected squares have whole number area measurements, but not a square number measurement, do not have a common measure with the unit of length.

The author A. Szabȯ does not agree with this interpretation. After a lengthy philological discussion of the concepts *dynamis* (square value of a rectangle) and *tetragonismos* (transformation into a square of equal area) he discovers inconsistencies in the terminology of Plato, who was of course not a trained mathematician. He writes:[3]

> Above all, I cannot agree when, based on the mathematical passage in the dialogue Theaetetus, it is claimed that Theaetetus is owed the exact definitions *measurable, squared measurable*, *rational* and *irrational* …. Nor can I agree when it is claimed that, at the end of our mathematical passage, the statement of Theaetetus (see above) is expressed shortly but very clearly.

Later, Szabȯ says that one cannot attribute any new mathematical concepts or terms for such sentences to the Platonic Theaitetos. Theaitetos is, so to speak, only a Platonic conversation partner, just as the mathematician *Timaios* from Lokri probably did not exist. In his book *The exact thinking of the Greeks* (p. 24), K. Reidemeister was the first in 1949 to express the suspicion that

> the mathematician Theaitetos is only a legend that has crystallized around the Theaitet[os] of the Platonic dialogue.

[3] Szabȯ, A.: Anfänge der griechischen Mathematik, Oldenbourg (1969), p. 97.

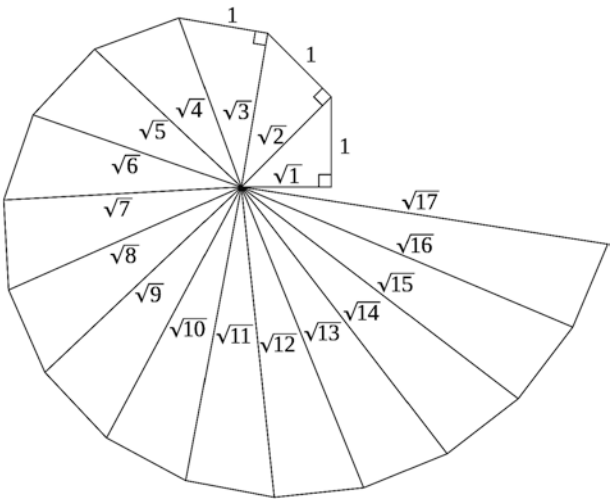

Fig. 6.5 Root spiral after Theodoros

From the fact that the Theaitetos theorem was possibly already known to the Pythagoreans, one cannot deny his existence; but the much-vaunted shaking of Pythagorean philosophy by the discovery of the irrational probably did not take place.

It is unclear why Theodoros stopped at the area of 17. The consideration of the so-called *root spiral* (Fig. 6.5), which would intersect at $\sqrt{18}$, is probably not sufficient explanation. B.L. van der *Waerden* has set out an elaborate interpretation of the root spiral in a series of articles.[4]

It is not even documented that Theodoros came to Athens. Many authors assume that he taught Plato before the death of Socrates, for this he must have visited the Academy. If Theodoros was not in Athens, Plato probably met him on one of his Sicily-Mediterranean journeys after the death of Socrates.

6.6 Theaitetos of Athens

Theaitetos (ca. 415–369) – if he even existed – was one of the few teachers of the Academy born in Athens. In addition to the Platonic dialogue named after him, in which he is referred as a student of Theodoros of Kyrene, there is a scholion[5] which speaks for his existence. It is found in the early Euklid edition of Commandino and attests that the sentences [Euklid X, 9–10] are from Theaitetos. The source Suidas notes:

[4] Van der Waerden B. L.: Die Arithmetik der Pythagoreer, Sammelband Becker (1965).

[5] A scholion is a remark added later to a manuscript, which often provides valuable information.

> Theaitetos from Athens, astronomer, philosopher, student of Socrates, taught in Herakleia. He was the first to write *about the famous 5 solids* . He lived after the Peloponnesian War.

A scholion to Euklid [XIII, 1] adds:

> In this book, it is the 13th, the 5 Platonic solids are described, but 3 of them are not his, because they were found by the Pythagoreans, namely the cube, the tetrahedron and the dodecahedron, while the octahedron and the icosahedron are due to Theaitetos. They were called Platonic because they are discussed in the Timaios.

In her dissertation, the mathematician E. Sachs[6] has shown that Theaitetos is actually the author of Book XIII of the *Elements*. An overview of mathematics at the academy of Plato gives Fowler (1987).

O. Neugebauer comments on Plato's role at the Academy as follows (quoted after Kedrovskij[7]):

> It seems to me obviously that Plato's role has been greatly exaggerated. His own direct contribution to mathematical knowledge is apparently zero. That in the course of a short time mathematicians of such stature as Eudoxus belonged to his circle is no evidence of Plato's influence on mathematical research. The exclusively elementary character of the examples of mathematical reasoning cited by Plato and Aristotle does not confirm the hypothesis that Theaetetus or Eudoxus had learned anything from Plato.

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[6] Sachs E.: Die fünf platonischen Körper, Philol. Unters. Heft 24, Berlin 1917.

[7] Kedrovskij O.I.: Wechselbeziehungen zwischen Philosophie und Mathematik im geschichtlichen Entwicklungsprozess, Leipzig 1984, p. 57.

Plato

7

Plato said: "God is a geometer." Jacobi changed this to: "God is an arithmetician." Then came Kronecker and coined the unforgettable phrase: "God created the natural numbers, everything else is human work" (Felix Klein).

ἀριθμον εἶναι τὴν οὐσίαν ἁπάντων (Number is the essence of all things) (Platon Metaph. 987A)

In the Peloponnesian War against Sparta (431–404), Athens suffered a severe defeat. Around 429 BC, there was a devastating epidemic, which was long believed to be the plague; however, newer research suggests typhus. In this difficult time, Plato was born into an aristocratic Athenian family in 427 BC (Fig. 7.1) and received a careful education for his future as a politician. His father *Ariston* and his mother *Periktione* both came from aristocratic families; Plato had two brothers Glaucon and Adeimantus and a sister Potone. Looking back on his youth, he wrote in the Seventh Letter (Ep. VII, 324b) why he withdrew from politics.

> When I was young, it happened to me as it often does to others, that I wanted to get involved in politics as soon as I turned 18. But then events took place that affected the city's political situation, namely the following. […] There was a coup, […] 30 men from the oligarchy took over the leadership of the entire state; among these were some of my relatives and acquaintances of mine, and they at once invited me to share in their doings […]. I believed, that they would lead the state from an unjust life to a just one and govern accordingly. […] So I watched them very closely to see what they would do. And seeing, as I did, that in quite a short time they made the former government seem by comparison something precious as gold – for among other things they tried to send a friend of mine, the aged Socrates, whom I should scarcely scruple to describe as the most upright man of that day ... by force to execution […]. Since I had to witness all this, and many other not insignificant things of this sort, I felt disgusted and withdrew from that vile activity.

After the victory over Athens, Sparta had set up a regime of 30 men; among these were his uncles Critias and Charmides, who offered him collaboration. These

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_7

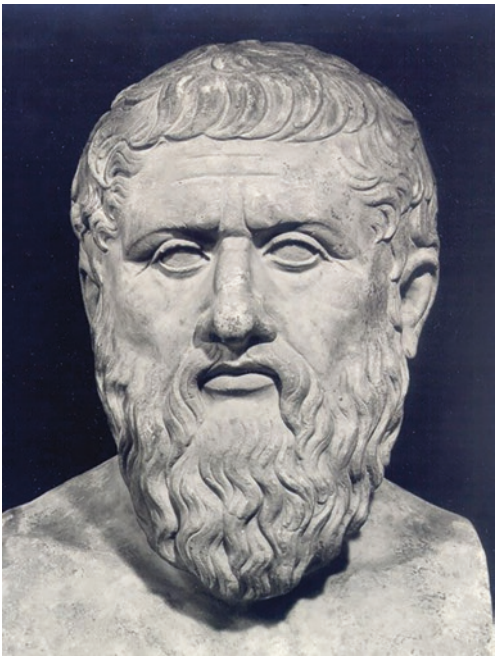

Fig. 7.1 Plato (Wikimedia Commons)

thirty exercised a reign of terror that claimed the lives of 1500 Athenians. This led Plato to the conviction that a meaningful political career was not possible in the face of the increasingly deteriorating morals. Only philosophy was able to restore justice in public and private life. After meeting Socrates, he decided to join him as a student for a longer period of time (Diogenes Laertios III, 6). The meeting with Socrates was so decisive for his life that he saw himself as the embodiment of Socrates. In the Second Letter (Ep. II, 314c) he wrote later *exaggeratedly* joking:

> I myself have written nothing about philosophy. All writings that are referred to as mine are works of Socrates, *who has become young and beautiful*.

As his role model Socrates was sentenced to death in 399 BC, he saw no possibility in Athens to meaningfully engage in politics. He had his own design of the *ideal state* in mind. He left Athens, first to Megara, to be mathematically instructed there by *Euklid of Megara*. His journey led him further to the Greek colonies of the Mediterranean, where he made the important acquaintance with the most important Pythagoreans. He met the famous and politically important Pythagorean *Archytas* of Tarent and *Theodoros* of Cyrene (North Africa).

In 389/388 Plato undertook his first journey to Sicily; in Syracuse he met the tyrant Dionysius I, whom he knew from his training with Socrates. He formed a close friendship with Dionysius' brother-in-law and son-in-law Dion, who became an ardent follower of Plato. It is uncertain what aroused Dionysius' displeasure;

anyway, Plato was not allowed to leave. However, he was later released and could return to Athens.

After his return, in 387 BC – as an Athenian citizen – Plato bought a piece of land at the Academia (Akademeia) called the Grove of the Attic Hero Academus in the northwest of Athens, on which he built his philosophical and scientific school. The goal of his life's work was to inspire young, gifted people with enthusiasm for philosophy, to train their minds in dialectical education, and to prepare them for a political career in the knowledge of truth, goodness and justice. Soon the most respected scientists gathered at the Academy, as the school was called: *Archytas* from Tarentum, *Theaitetos* from Athens, *Eudoxos* from Cnidus, *Amyklas* from Heraclea and also *Aristotle.*

Plato himself was not a mathematician; his influence on mathematics through his philosophy and the formation of a curriculum was immense. Proclus writes about him in his commentary on Euclid:

> *He caused a very great advance in mathematics in general and geometry in particular through his enthusiasm for these [subjects], which is evident from the way his books are filled with mathematical examples and he tries to inspire admiration in those who pursue philosophy everywhere.*

In addition to the teaching activities, symposia were also held at the academy, to which guests also had access (Fig. 7.2). So the important Athenian politician and general Timotheus reports that *he had been invited by Plato and had found that the food and drink were simple, but the conversations were rich.*

Despite the bad experiences on the first Sicilian journey, he followed an invitation that had been made by the son and successor of the tyrant, Dionysios II, at the instigation of Plato's friend Dion. Dionysios I had won the fight for Sicily against

Fig. 7.2 Symposium depiction—vase painting (Wikimedia Commons)

Carthage and had raised himself to sole ruler. Plato hoped that, in cooperation with Dion, he would be able to bring his political ideas to the young ruler and, if possible, to set up a state according to the ideal of the philosopher's rule. But this failed because Dionysios II had other plans.

In 361 BC, Plato traveled to Sicily for the third time. Dionysius II had Dion arrested and Plato tried to free him. This led Plato to be suspected of planning a political overthrow and he was himself taken prisoner. From this dangerous situation, he was saved by Archytas, who intervened from Tarentum and made his return to Athens possible in the summer of 360.

Plato spent his last years teaching and researching at the Academy. According to Cicero, he died at the age of 81, in 347 BC, and was buried near the Academy. To the disappointment of Aristotle, Plato's nephew Speusippus was elected as his successor at the Academy.

That Plato had the inscription *No one ignorant of geometry enter* placed at the entrance of the Academy, as J. Tzetzes reports in the Chiliads [VIII, 974], is probably an invention from a later time. D. Fowler[1] (p. 201) established that the motto has the usual form to prevent a non-initiated person from entering the temple. The word *ageometretos* he found with Aristotle [Anal. Post. 77b]:

> Therefore, one must not discuss geometry with those who are ignorant of geometry (ageometretos), because in such a company an unsound argument will go unnoticed.

The earliest complete biography dates only from the middle of the 2nd century AD, that is, it was written only 500 years after Plato's death by *Apuleius* from Madaura (today Algeria). So all biographical material is lost; this had the consequence that the later biographers were dependent on unreliable facts. A detailed report on this is by Klaus Döring[2].

7.1 Plato's Most Beautiful Triangles

After assigning the 5 Platonic solids to the four elements and universe, Plato tries in the *Timaios* [53b–55c] to explain the regular polygons appearing as surfaces of the solids by means of special triangles.

> But all triangles go back to two, each of which has a right and two acute angles: one in which two sides are equal […] and the other with two unequal sides […]. Among these two triangles, the isosceles one only has one type, but the unequal one has countless.

Plato here means that all isosceles right-angled triangles are similar, in contrast to the non-isosceles. Among the latter, he makes a selection

[1] Fowler D.H.: The Mathematics of Plato's Academy, A New Reconstruction, Oxford Science 1987, p. 201.

[2] Döring K.: Die antike biographische Tradition, pp. 13–17, in the Platon-Handbuch.

> We therefore want to have chosen two triangles: one isosceles and the other in which the square of the larger cathetus is three times that of the smaller one.

Plato here means the right-angled triangle with the angles (30°; 60°; 90°), which is similar to the triangle with the sides $\left(1; \sqrt{3}; 2\right)$ (Fig. 7.3). For the composition of the isosceles triangles, he writes:

> The basic element of the same is (as said) the triangle, whose hypotenuse is twice the length of the smaller leg. If now two such triangles are composed to form a rectangle, so that their hypotenuse becomes the diagonal of the latter, and this is repeated twice more, so that all diagonals and the smaller legs meet in one point, then a single isosceles triangle results from six such triangles, the center of which is precisely that point.

If two of these triangles are placed together at their hypotenuse, a deltoid results; three of these deltoids form an equilateral triangle. With these triangles, the surface of the tetrahedron, octahedron and icosahedron is composed.

For the squares as surface figures of the cube, Plato needs the right-angled isosceles triangle. Two of these triangles result, assembled at the hypotenuse, in a square; four of these small squares provide a large one (Fig. 7.4).

However, Plato does not succeed in decomposing the pentagon (required for the dodecahedron) into congruent, right-angled triangles of either of the two types mentioned above. A decomposition of the pentagon into 30 right-angled triangles is possible; however, ten triangles each come from three different congruence classes. The number 30 of the pentagon can be found in Albino (*Introd. doctrinae Plat.*, 13), who writes:

> For the universe, God made use of the dodecahedron. Therefore, we see in the sky the shapes of 12 animals in the zodiac […] And almost like the dodecahedron, which consists of 12 pentagons, each of which can be decomposed into 5 triangles, each of which in turn consists of six triangles, 360 parts result, which can also be found in the zodiac.

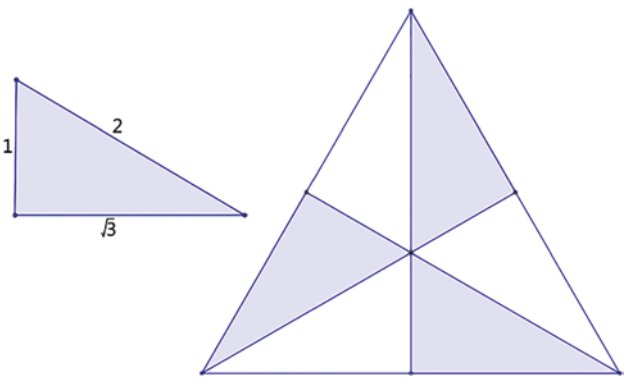

Fig. 7.3 Plato's beautiful triangle no. 1

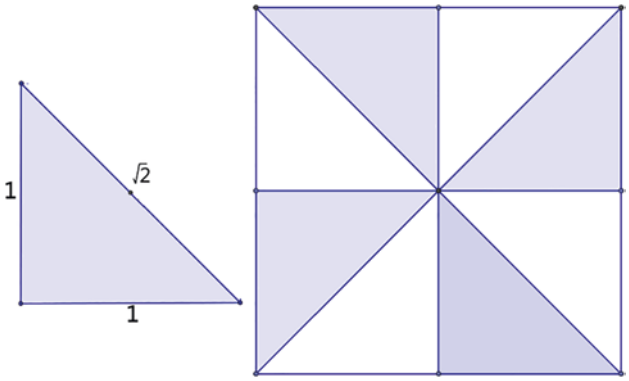

Fig. 7.4 Plato's beautiful triangle no. 2

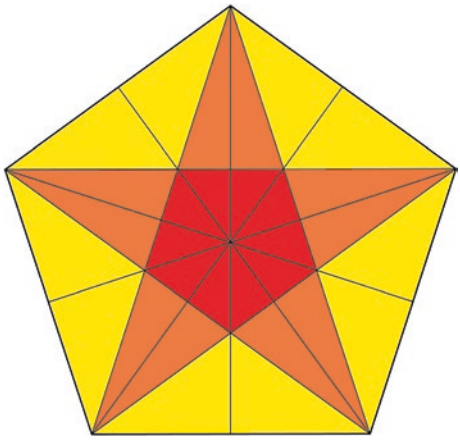

Fig. 7.5 Pentagram decomposition into right-angled triangles

The diagonals in the regular pentagon form a star pentagon (*Pentagramm* called), which was probably a secret or recognition sign of the Pythagoreans (Fig. 7.5). The sign was called by the Pythagoreans ὑγιεία (*Hygieia* = health); the desire for health was an obligatory part of the Pythagorean greeting.

7.2 From the Book Menon

Menon 87A

The early book Menon (written around 388, after the first Sicily trip) contains the famous dialogue in which Socrates explains to a slave the doubling of the square area. In addition to this passage, there is another passage [Menon 86E–87 C],

which is not quite easy to interpret. This passage is probably the oldest (extant) Greek text that leads to a problem in higher mathematics. Plato has Socrates speak:

> For example, if someone asks about a figure, whether it is possible to inscribe this triangular figure in a circle? […]. If the figure is such that, if one extends the given line of the same, the space one cuts off is as large as the space added by the extension.

According to O. Becker, the passage can be interpreted as follows: An isosceles triangle ΔABC, whose area is given by a rectangle $ADCE$, is to be inscribed in a circle (Fig. 7.6). The base AD of the rectangle, which is also the symmetry axis of the triangle, is extended to the point F, in such a way that the newly created rectangle $DFGC$ is similar to the given rectangle. The line AF is then a diameter of the sought circumcircle of ΔABC.

Analysis: If the construction is carried out according to the description, the lines BC and AF form a pair of chords that intersects in the circle and it holds (theorem of chords):

$$|CD||DB| = |AD||DF|$$

For the right-angled triangle ΔAFC it follows from the altitude theorem

$$|CD|^2 = |AD||DF| \Rightarrow \frac{|CD|}{|DF|} = \frac{|AD|}{|CD|}$$

This proportion shows the similarity of the two rectangles $ADCE$ and $DFGC$. If the extension of AD is not such that $DFGC$ is similar, then there is no solution with the given method.

The algebraic solution of the problem leads to an equation of the fourth degree. Let a^2 be the area of the given rectangle $ADCE$ and thus the area of the triangle ABC. Setting $|AF| = d$, $|AD| = x$ and $|CD| = y$,, we get from the above equations

$$y^2 = x(d - x) \therefore xy = a^2$$

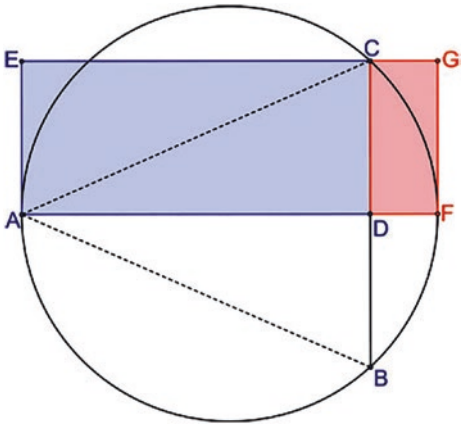

Fig. 7.6 Figure for Menon 87A

Inserting the second condition into the first one yields with $y = \frac{a^2}{x}$

$$\left(\frac{a^2}{x}\right)^2 = x(d-x) \Rightarrow a^4 = dx^3 - x^4$$

Heath (I, p. 301) gives a similar solution for the enrollment at (Fig. 7.7):

Let AB be the diameter of the given circle; the rectangle $ADEC$ of the given area is placed so that the side AD lies on the diameter. In point A a coordinate system is set up. In the corner point E a ruler is placed; the line through E is to have the intersection points R and T with the coordinate axes and P with the circle. The line through E is rotated until the segments $|RE| = |PT|$ are congruent (*Neusis* construction). The point P and its mirror point Q with respect to AT form with the point A the desired isosceles triangle ΔAPQ equal-area to rectangle $ADEC$.

Proof: According to the assumption, $|RE| = |PT|$. Therefore, the triangles ΔRCE and ΔPMT are congruent, since they match in one line and in two adjacent angles. Thus, the other sides are also congruent, that is $|PM| = |RC|$ and $|AD| = |CE| = |MT|$, with M being the foot of the perpendicular from P dropped upon AT. From this follows $|AM| = |AD| + |DM| = |MT| + |DM| = |DT|$. With the similarity of the triangles ΔEDT and ΔPMT, the proportion follows

$$\frac{|AM|}{|AD|} = \frac{|DT|}{|MT|} = \frac{|ED|}{|PM|} \Rightarrow |AM||PM| = |AD||ED|$$

This shows the desired area equality of rectangle $ADEC$ and triangle APQ. In this interpretation, Plato would show how to transfer the problem of inscribing a figure to the area layout of a similar figure.

Menon 84E

Socrates here develops, in a question-and-answer conversation with a slave who has no mathematical training, the procedure for doubling the area of a given

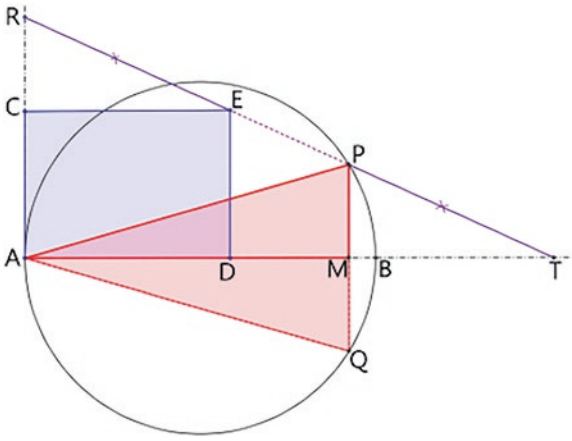

Fig. 7.7 Alternative figure to Menon 87A

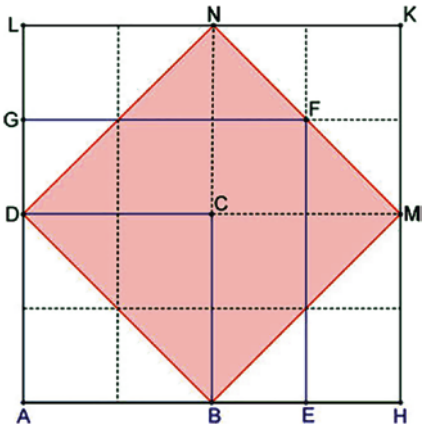

Fig. 7.8 Figure for Meno 84E

square *ABCD* with the side length 2 units (Fig. 7.8). A doubling of the side length (to 4 units) is clearly seen to result in a fourfold content (square *AHKL*). The choice of square *AEFG* with the middle value (3 units) also does not provide a solution. Only the choice of square *DBMN* upon the diagonal *BD* of the original square leads to the desired result. Since the slave did not know the result, Socrates attributes this to the *recollection of the soul* (anamnesis) of the slave.

Socrates' attitude towards mathematics is ambivalent: on the one hand he mocks mathematical activities, on the other hand he admits that mathematical knowledge consists of *objects of eternal being*. The following dialogue between Glaucon and Socrates can be found in Republic (Politeia 527A):

> Socrates: Your expressions [those of geometry] are most ridiculous and forced; for as if you were setting something in motion and trying to achieve a real effect, you choose all your expressions, such as making square (squaring), stretching out (oblonging), adding on (adding), and all the other words you use; but in fact, the true purpose of all this knowledge is nothing other than pure understanding.
>
> Glaucon: Entirely so.
>
> Socrates: Then we must come to an understanding about this, mustn't we?
>
> Glaucon: About what?
>
> Socrates: That this understanding goes back to what is always existing, and not to what comes into being and passes away.
>
> Glaucon: There is no need for that, for the geometric understanding always has to do with what exists.
>
> Socrates: So then, my good man, there would be a power in it that draws the soul towards truth and creates a philosophical way of thinking, to the extent that we then direct what we now wrongly direct downwards, upwards.

In the work *Georgics*, which was probably written at the same time, there is a passage that counts mathematics among the easier sciences:

> But there are also other arts which achieve their goal entirely through speech and make very little or no work at all necessary, for example arithmetic, geometry, board games, and many others; in some of them, speech and action are kept in pretty much balance.

7.3 Platonic Solids

A Platonic solid (Fig. 7.9) is a convex polyhedron whose surface consists of congruent, regular polygons (of the same number of vertices). The surface of the tetrahedron (4), octahedron (8), and icosahedron (20) consists of equilateral triangles. The surface of the cube contains 6 squares; the surface of the dodecahedron 12 pentagons. The 5 solids were named after Plato, because they are mentioned in his dialogue [Timaeus 54, 55]; he defines the regular polyhedron

> as a solid by means of which the whole (surrounding) sphere is divided into equal and similar parts.

The ancient Pythagoreans were initially only aware of four bodies (without an icosahedron). Four of these bodies (without a dodecahedron) were assigned to the elements fire, earth, air, water. Plato writes in Timaeus 55b

> Let us rather distribute the four shapes […] among the four so-called elements fire, earth, water and air […]. But since there is also a fifth type of composition of corresponding properties, God rather used this for the world as a whole, when he gave it its adornment (κόσμος = adornment, world).

In search of what can be associated with the dodecahedron, Plato comes up with the idea of the universe, since the 12 surfaces correspond to the 12 constellations of the zodiac signs. He lets Socrates speak in the dialogue [Phaedo 110b]:

> First of all, my friend, it has been said that the earth, seen from outside, looks like a ball made of 12 leather patches, painted with different colors that are familiar to us like the samples that our painters use. Up to the top the whole earth shows colors that are brighter and purer than those.

In the Middle Ages, the dodecahedron symbolized the so-called *Fifth Element* (quinta essentia). Figure 7.10 shows the five elements from the work *Harmonices Mundi* by J. Kepler. At first, Kepler tried to describe the then-current planetary system (with 6 planets) by inscribing the Platonic solids between the planetary orbits.

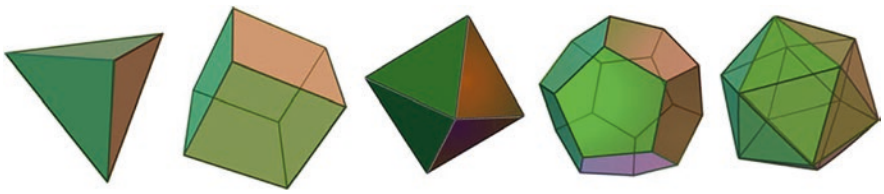

Fig. 7.9 The five Platonic solids (Wikimedia Commons)

Fig. 7.10 Platonic solids in Kepler's *Harmony of the World* (1619), painted by Kepler

Euklid claims in the book (XIII, 18a) that only the five bodies mentioned above are composed of similar, congruent polygons. He implicitly assumes here that the same number of surfaces come together in all corners. This condition is not met, for example, in the double tetrahedron or in the double five-sided pyramid. These bodies arise when two tetrahedrons or two 5-sided pyramids are combined (Fig. 7.11). The surface of the double tetrahedron thus comprises 6 equilateral triangles; with $E = 5$ corners, $F = 6$ surfaces and $K = 9$ edges, it also satisfies the *Euler polyhedron formula*

$$E + F - K = 2$$

Another implicit assumption of Euclid's is hidden in definitions 9/10 of book XI: Equal and similar are bodies that are surrounded by similar plane surfaces in the same number and size. Two polyhedrons are thus congruent if corresponding surfaces of their surface are congruent. Here, the convexity of the polyhedrons is assumed. Figure 7.12 shows a cuboid to which a four-sided pyramid has been added or cut out.

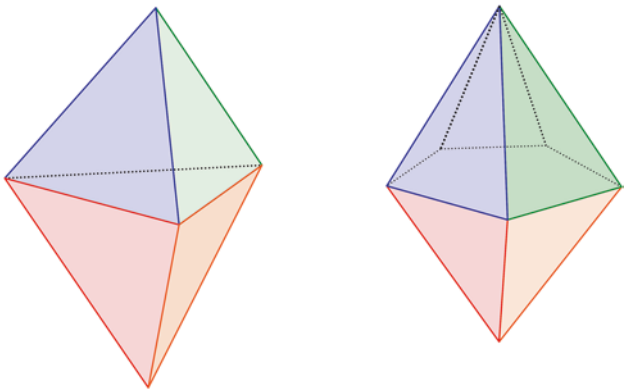

Fig. 7.11 Image for Euklid XIII, 18a

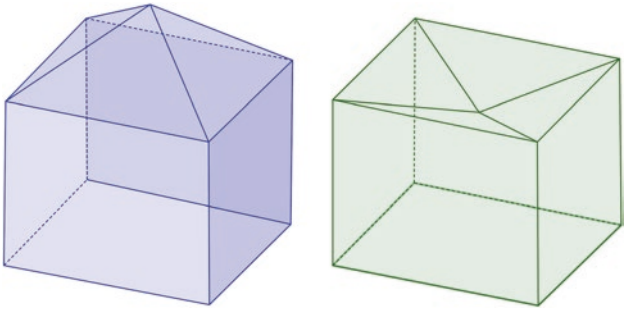

Fig. 7.12 Image for Euclid definition XI, 9

The following table results for the Platonic solids

	Vertices E	Surfaces F	Edges K
Tetrahedron	4	4	6
Cube	8	6	12
Octahedron	6	8	12
Icosahedron	12	20	30
Dodecahedron	20	12	30

There are no other convex polyhedra, which are surrounded by congruent, regular figures and at which an equal number of surfaces meet at each corner, in addition to the 5 Platonic solids. It is striking that the pairs (cube, octahedron) and (icosahedron, dodecahedron) have the same number of edges and that the number of corners E of the first is equal to the number of edges K of the second. Such bodies are called *dual;* for example, an octahedron is obtained by connecting the centers of all the faces of the cube (Fig. 7.13).

Proof [Euclid XIII, 18a]:

A corner (in space) cannot be erected from 2 triangles or other plane surfaces; however, from 3 triangles of the pyramid (tetrahedron), from 4 of the octahedron, or 5 of the icosahedron. There cannot be a corner from 6 and more triangles, since the sum of the interior angles would then be at least $6 \cdot \frac{2}{3}R = 4R$, which is not possible according to [Euclid XI, 21]. A cube corner encompasses 3 squares; 4 squares would result in 4R. A dodecahedron corner encompasses 3 (regular) pentagons; 4 would result in the sum $4 \cdot \frac{6}{5}R > 4R$.

Modern proof: First, the interior angle of an n-sided regular polygon is determined to $\frac{n-2}{n}\pi$. If surfaces meet in a convex corner r, then (in radians)

$$r\frac{n-2}{n}\pi \leq 2\pi \Rightarrow rn \leq 2(r+n)$$

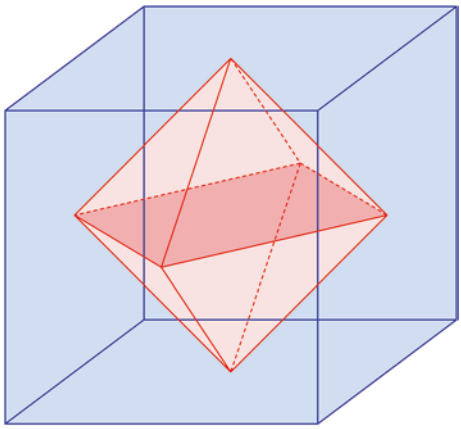

Fig. 7.13 Duality of Platonic solids

Since a polygon must have at least 3 sides, it is certainly $n \geq 3$. Furthermore, at least 3 surfaces must meet at a vertex of a polyhedron; thus $r \geq 3$. Substituting these values shows that 5 integer solutions exist:

- $(n, r) = (3,3) \rightarrow$ Tetrahedron
- $(n, r) = (4,3) \rightarrow$ Cube
- $(n, r) = (3,4) \rightarrow$ Octahedron
- $(n, r) = (5,3) \rightarrow$ Dodecahedron
- $(n, r) = (3,5) \rightarrow$ Icosahedron.

Proof with polyhedron formula

The number of Platonic solids can be easily determined using the polyhedron formula. The regular polyhedron has F faces, each of which is a regular polygon with n corners. At each corner, r edges should meet. Count the edges on the one hand according to the surfaces and take into account that each edge belongs to 2 surfaces, then follows

$$nF = 2K$$

If you count the edges on the other hand according to the corners and take into account that each edge belongs to 2 corners, you get $rE = 2K$.

Solving for E or F and substituting into the polyhedron formula gives

$$\frac{2K}{r} + \frac{2K}{n} - K = 2 \Rightarrow \frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{K}$$

As above, $n \geq 3$ and $r \geq 3$ apply. It is easy to see that not both parameters n, r can be greater than 3, otherwise the right side is not greater than $\frac{1}{2}$. Setting $n = 3$, thus follows

$$\frac{1}{r} - \frac{1}{6} = \frac{1}{K}$$

Fig. 7.14 Five of the 13 Archimedean solids

An integer solution only results for $3 \leq r \leq 5$. These 3 values provide $K \in \{6; 12; 30\}$. These edge numbers correspond exactly to the polyhedra given above. In the case of $r = 3$, the procedure is analogous

$$\frac{1}{n} - \frac{1}{6} = \frac{1}{K}$$

This provides the inequality $3 \leq n \leq 5$. These 3 values also lead to $K \in \{6; 12; 30\}$. These edge numbers characterize the given polyhedron.

All Platonic solids have an inscribed sphere, as Euclid proves in Book XIII. If R is the radius of the circumscribed sphere, then for the edge length of the inscribed body according to Euclid

Body	Edge length
Tetrahedron	$\frac{2}{3}R\sqrt{6}$
Cube	$\frac{2}{3}R\sqrt{3}$
Octahedron	$R\sqrt{2}$
Icosahedron	$\frac{1}{5}R\sqrt{10\left(5-\sqrt{5}\right)}$
Dodecahedron	$\frac{1}{3}R\left(\sqrt{15}-\sqrt{3}\right)$

Outlook

If one drops the condition that all polygons of the surface must be congruent, one obtains 13 semi-regular bodies, which are attributed to Pappus of Archimedes. A detailed treatment can already be found with Kepler.

Figure 7.14 shows five of the 13 Archimedean solids which are created by cutting off corners from the Platonic ones. A detailed description can be found with Aumann[3].

The fifth body pictured is an icosahedral truncation and resembles a classical football. Since it consists of 12 regular pentagons and 20 hexagons, its surface area is $F = 32$. It also satisfies the Euler polyhedral formula, since it has $E = 60$ vertices and $K = 90$ edges.

[3] Aumann G.: Archimedes—Mathematics in Moving Times, Wissenschaftliche Buchgesellschaft 2013, pp. 181–202.

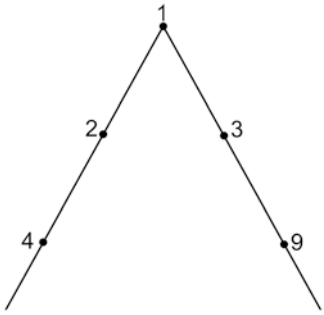

Fig. 7.15 Lambda-Figur nach Platon

7.4 Plato's Lambda

In the book *Timaios* 35A, God mixes body and soul substance:

> First he took away a part of the whole [unity], then the double of the same, then the third the one and a half times of the second part, the fourth the double of the second, the fifth the triple of the third, the sixth the eightfold of the first and the seventh the twenty-sevenfold of the first ….

This sequence of numbers is noted in the figure next to it, which is referred to as *Plato's Lambda* because of its similarity to the letter (Fig. 7.15). It can be found in Theon von Smyrna[4].

The report of the Timaios is continued later:

> But this […] structure he divided lengthwise into two parts, crossed them in the middle so that they formed the shape of a Chi (χ), and then bent each of them into a circle […]. But the inner [space] he divided sixfold and thus divided it into seven unequal circles, depending on the intervals of the double and the triple.

The formation of circles in this figure is most often interpreted as inscribing a regular hexagon into the lambda figure; this completes the figure to the equilateral triangle highly esteemed by Plato. Since the number of elements is equal to the *holy* number 10 and the figure consists of 4 rows, it can also be referred to as a tetractys (Fig. 7.16). The Timaeus text does not explain how the resulting gaps are to be filled. According to the text, the distances of the connecting elements $1\frac{1}{3}$ or $1\frac{1}{8}$ are to be $\frac{4}{3}$ or $\frac{9}{8}$ has not yet been found. However, the gaps of the middle hexagon can be filled so that the central $\frac{4}{3}$ or $\frac{9}{8}$ has not yet been found. However, the gaps of the middle hexagon can be filled so that the central *six* is the geometric mean of the opposite neighbors:

[4] Theonis Smyrnaei philosophi Platonici expositio rerum mathematicarum ad legendum Platonem utilium, Ed. E. Hiller, Leipzig 1878, Chap. 38.

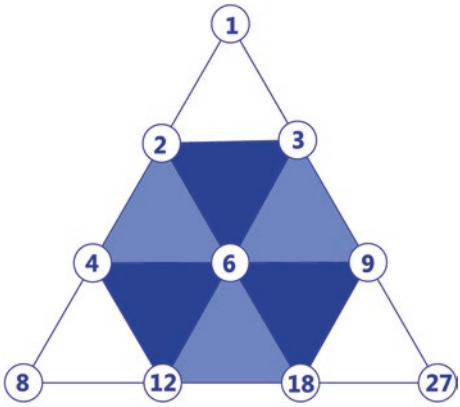

Fig. 7.16 Extended lambda figure

$$6 = \sqrt{2 \cdot 18} = \sqrt{3 \cdot 12} = \sqrt{4 \cdot 9}$$

The ratios of adjacent numbers in the base line are constant here $\frac{2}{3}$

$$\frac{8}{12} = \frac{12}{18} = \frac{18}{27} = \frac{2}{3}$$

Thus, all elements of the triangle sides (except the corners) represent the geometric mean of their two neighbors. This gives the triangle figure a special symmetry. The aforementioned ratio also occurs when the number sum (= 36) of the triangle corners is related to the sum in the hexagon (= 54)

$$\frac{36}{54} = \frac{2}{3}$$

The symmetry of the figure also results in the number sum (= 54) of the lambda figure being equal to the sum in the hexagon (= 54).

There are still two numbers that Plato explicitly mentions in the book Politeia (Republic), the meaning of which is unclear. The first is the famous *wedding number* 5040 [Politeia 546A–D], which Plato claims has the numbers 1 to 10 as divisors and the number of all proper divisors is $60 - 1$. The prime factorization is

$$5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 7!$$

This results in the number of all divisors of 60, the sum of the proper divisors is actually 59, as claimed. The second mysterious number is the so-called *tyrant number* or number of the beast [Politeia 587C]. There it is asked how far the tyrant soul deviates from the reasonable, royal soul. Plato has Socrates speak:

> The oligarch, again, is three times thrice removed from the king, and thus we get the formula 3 x 3, which is the number of a surface, representing the shadow which is the tyrant's pleasure, and if you like to cube this *number of the beast*, you will find that the measure of the difference amounts to 729; the king is 729 times more happy than the tyrant.

This is about the number

$$\left(3^2\right)^3 = 3^6 = 729$$

Plato does not explain what special meaning the number should have, which is only divisible by powers of three.

7.5 The Role of Mathematics in Plato

It can be assumed that Plato has received an education in which mathematics has played only a subordinate role. On his travels he had longer encounters with leading Pythagoreans, so that their mathematical knowledge was familiar to him. Mere reception of what was heard was not Plato's role. Based on his writings *Gorgias, Menon,* and *Theaitetos* it can be seen that he penetrates to a certain extent into mathematics and finally makes this the basis and the standard of all science.

The importance of mathematics must also relate to the teaching or education. The philosophically educated rulers should not be ignorant of geometry [Politeia 527C]. In the books *Politeia* (Republic) and *Nomoi* (Laws) the philosophical education of the youth in arithmetic, geometry, stereometry and astronomy is demanded, which certainly led to the fact that mathematics – as our cultural heritage, although initially fought by Christianity – is still in our curriculum. But this education was only intended for free citizens. In the dialogue Laws [Nomoi 817E–818A] Plato lets the Athenian speak:

> There are also three subjects for the freeborn to learn: first of these the first is reckoning and arithmetic; the second is the art of measuring length and surface and solid; the third deals with the course of the stars, and how they naturally travel in relation to one another.

Aristoxenus and numerous others tell the following anecdote, which Aristotle experienced during his time at the Academy (quoted from O. Toeplitz[5]). Plato announces a lecture *On the Good* and numerous listeners come expectantly:

> Everyone seems to assume they would achieve some human goods, like wealth, health, strength, or even a wonderful happiness. But when the arguments about mathematics, numbers, geometry, and astronomy began, and *the basic principle of goodness is the One*, the surprise must have been general. Some lost interest in the subject, while others criticized it.

As we can see, Plato already posed a puzzle to his contemporaries. In addition to the idea of the Good, there are the ideas of Beauty, Justice, etc. Geometry is the prototype for these ideas. In the Seventh Letter (Epist. VII, 342B), Plato distinguishes four moments in every thing.

[5] Toeplitz O.: Mathematics and Ideology at Plato, in the anthology Becker, p. 61.

> The first moment is the name, the second is the concept definition expressed in language, the third is the image perceptible through the bodily senses, the fourth is the full intellectual knowledge.

Plato explains this with a mathematical example, the circle:

> The circle is, for example, a special predicated thing that just has the name that we just mentioned. The second of that thing, the […] definition would be: from its ends to the center, everywhere the same distance away—this would be the definition of that thing that has the name *round, circle, compass*. The third is the physical image that falls into the outer senses, e.g. from the draftsman or the turner, which can be extinguished and destroyed again—events to which the archetype [the idea] of the circle itself […] is not subject, because it is something quite different and entirely distinct from it.

All ideas have the following characteristics: They are not empirically observable and they are imperishable. They are also only thoughts graspable or can be *guessed*. A beautiful flower represents the idea of beauty; the flower withers, but the idea of beauty remains. What we can see of the flower is its transience. What we perceive only shows the inadequacy and transience of the specimen flower.

The famous *Allegory of the Cave* [Politeia 514A] shows us that sense impressions can deceive us. The people who are chained up in the cave only see the shadows of the outside world and the flickering light of the cave. This is the realm of the perceptible. After they are freed, the people climb up and see – at first still painfully blinded by the increasing light – the images of the true world; they move from the stage of mere perception to the knowledge of the True. The allegory here symbolizes the arduous journey of the soul of the uneducated person to the true philosopher. In [Politeia 520] Socrates says to those who have made it out of the cave:

> You will see a thousand times better than those down there [in the cave] see and you will recognize each shadow image for what it is and what it is of, because you have seen the Beautiful, the Good, and the Just themselves.

For Plato, it is the task of philosophy to discover the true nature of things, which lies behind the external appearance, the constant change and flow of time. In this task, mathematics takes a central place, because mathematical knowledge is an outstanding example of knowledge that is independent of sensory experience and knowledge of the eternal and necessary truth. For Socrates in the famous Menon dialogue, the knowledge of the slave was a memory of a life before. For Plato, this is an indication that there is a true knowledge and understanding of the Eternal. There are thus truths in geometry that we have not learned through training or experience. This knowledge is part of the immutable, universal truth.

Plato describes the different stages of knowledge in the famous allegory of the divided line [Politeia 510A] (Fig. 7.17). The appreciation of mathematics is shown by the fact that he presents the objects of mathematics as a separate class of

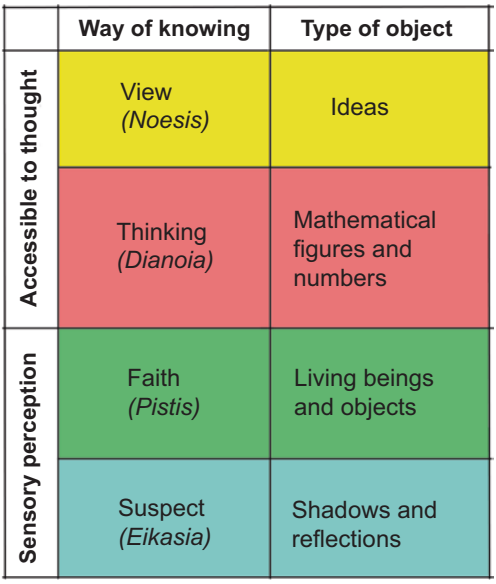

Fig. 7.17 Representation of the allegory of the divided line

objects of the mind, although he assigns them a lower degree of being and knowledge than the ideas themselves. A detailed analysis of the diagramm can be found in Lattmann[6].

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Aristotle and the Lyceum

8

Aristotle is one of the richest and most comprehensive scientific geniuses that has ever appeared,—a man to whom no time has an equal. [For he] has penetrated into the whole mass and all sides of the real universe and has subjected their wealth and dispersion to the concept; and most philosophical sciences owe their distinction, their beginning to him. (G.W.F. Hegel)

8.1 Life and Work of Aristotle

Aristotle (Ἀριστοτέλης) was born in 384 BC in Stageira (Macedonia). His father *Nikomachos* was the personal physician of King Amyntas III of Macedonia, at whose court in Pella Aristotle grew up. Since his father died when he was still a minor, Proxenus from Atarneus took over Aristotle (Fig. 8.1) as guardian. Proxenus let Aristotle experience a comprehensive education, at the end of which he was sent to study in Athens.

He came there in 367 BC as a seventeen-year-old and joined Plato's Academy, where he spent a total of 20 years. Since Plato undertook three trips to Sicily during this time, it can be assumed that Aristotle also took part in lectures by other members of the Academy, including the mathematician and astronomer *Eudoxos* of Knidos. He was friends with Plato, who was about 40 years older. His famous quote is:[1]

> Plato amicus, magis amica veritas. (Plato is a friend to me, but I am more friendly with the truth.)

[1] Vita Vulgata $ 9, quoted from: Düring I.: Aristotle in the ancient biographical tradition 1957.

Hegel G.W.F., quoted by Flashar H.: Aristoteles, Lehrer des Abendlandes, Chap. 10

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_8

Fig. 8.1 Aristotle (Wikimedia Commons)

According to Diogenes Laertios (III, 46), Aristotle belongs to the inner circle of 19 students (such as *Speusippos* and *Xenocrates*) around Plato. It can be assumed that Aristotle already gave his own lectures on rhetoric or dialectic at the Academy, even though the preserved works on this subject date from his late work. In doing so, he developed his own philosophical view, which criticizes Plato's theory of ideas; however, he maintained his personal appreciation, as the quote[2] (Fragment 673) shows:

> … the bad [among the philosophers] must not even praise him [Plato].

In the fragments he praises the virtue, Plato is

> … the only or first of the mortals, who by his own life and the method of his words made it clear how the virtuous man at the same time becomes a happy man.

After Plato's death, Aristotle left Athens in 347 BC; twelve years of wandering followed. One possible reason was his dissatisfaction that, contrary to expectation, Plato's nephew Speusippos took over the leadership of the Academy. According to Diogenes Laertius (V, 1), when Aristotle left suddenly, Plato is said to have commented: *Aristotle gave me a kick, like a foal to its mother.*

[2] Rose V. (ed.): Aristotelis qui ferebantur librorum fragmenta, Teubner Leipzig 1886.

In addition, there was an animosity in Athens directed against the Macedonian occupation, which also turned against him because of his descent. He followed an invitation from the tyrant Hermias, a former Plato student, who ruled in the city of Assos (near the island of Lesbos). In 341 he married Hermias's sister (or niece) *Pythias.* In Assos he also met Theophrastos from Eresos, who became his friend and later his collaborator. After Hermias was executed by the Persians in 342, he left Assos and settled in Theophrastos's hometown of Mytilene on Lesbos, where both began studying biology together.

The observations made in this richly varied bay *Kolpos Kalloni* represent the beginning of biology as a natural science; many of his scientific writings go back to the experiences made there. In his writings, Aristotle mentions more than 500 species of animals! A detailed description of these studies can be found in A. M. Leroi[3]. When C. Darwin received a copy of Aristotle's book *De partibus animalium* two months before his death (1882), he wrote to the translator:

> I already had a high opinion of Aristotle's merits, but I had no idea what a wonderful person he was. Linné and Cuvier were—in very different ways—my two gods, but compared to the old Aristotle they were just schoolboys.

In 343 Aristotle received an invitation from Philip II to the royal court, to be the tutor of the 13-year-old son Alexander. The lessons ended after three years, when the 16-year-old Alexander took over the regency. Aristotle had a copy of the *Iliad* made for Alexander, which the latter carried with him on all his conquests. In order to break the resistance of the rest of Greece to Macedonian domination, Alexander had Thebes completely destroyed in 335 and all the inhabitants sold into slavery. He then had free hand in Greece and could start the Persian campaign in 334.

As a result of the Macedonian hegemony, Aristotle could return to Athens in 335 BC. When the position of Academy director became vacant after the death of Speusippus in 339 BC, he again did not come into play. Since Aristotle did not have citizenship and thus no building rights in Athens, he was assigned a plot of land by Antipater, the Macedonian governor, and founded a public gymnasium together with Theophrastus. On the site was a grove dedicated to Apollo Lykeios, so the school was named *Lykeion* (λῦκειον, Latin *lyceum*). Since the school had a large public colonnade (stoa), it was also called *Peripatos* (περίπᾶτος, walk, colonnade), and the scholars working there were called *Peripatetics.* Figure 8.2 shows the prototype of a colonnade in Athens, which was founded by King Attalus.

In the following twelve years he was able to expand the school and founded a large universal library, which earned him recognition throughout Greece. Aristotle employed a whole staff of scientific collaborators, such as *Theophrastos* for natural science, *Eudemos* for mathematics, *Menon* for medicine and others. Also Theophrastos, who for 36 years was his successor at the Lykeion, was successful

[3] Leroi A. M.: The Lagoon or How Aristotle Invented Science, Theiss 2017.

Fig. 8.2 Example of a colonnade (stoa, Wikimedia Commons)

in recruiting high-ranking scientists to his institute. These were *Herakleides* of Pontus for astronomy, *Straton* of Lampsakos for physics and *Dikaiarchos* for geography. The latter science took on enormous momentum through the knowledge gained through Alexander's conquests in the Orient. In his will, Theophrastos stipulated that all maps produced should be publicly displayed in the colonnade.

We owe two compilations to the scientific work of the Lykeion, which have greatly promoted our understanding of Greek science:

- The first collection is the famous *History of Mathematics* by Eudemus, which has been partially preserved by Proclus (Sect. 24.1).
- The second volume, to which we owe our knowledge of early philosophy, is Theophrastus' book Φυσικῶν δόξαι *(Doctrines of the Physicists).* This work is the only surviving summary of the opinions and statements of the ancient philosophers; it was done at a time when the tradition was still alive. It is the merit of Hermann Diels to have traced and arranged the sources of this tradition. He was able to show that all the quotations and fragments go back to the aforementioned writings of Theophrastus. This work continued to have an effect in antiquity and became an important source for later writings. His records were published by Diels in his first volume, *Doxographi Graeci* (1877).

With the death of Alexander in 323 BC, the Macedonian empire collapsed. In Athens and other Greek cities, an anti-Macedonian sentiment arose. Even Aristotle

was attacked in Athens; he was even threatened with a trial for undermining piety (asebie). Therefore, he fled from Athens; as Aelian reports in *Varia historica*, Aristotle left the city with the remark:

> He could not allow the Athenians to commit a second offence against philosophy [after the death sentence against Socrates].

He went into exile and retired to his parents' house in Chalkis (on Euboea), where he died in October 322 BC at the age of 62. After the death of his wife Pythias, Aristotle found a life partner in Herpyllis, his favorite slave. She was probably the mother of his son Nikomachos; both were provided for financially in his will. The will of Aristotle, as recorded by Diogenes Laertius (V 1,11–16), has literary and legal quality.

Not preserved are the literary writings of Aristotle, which were still accessible to Cicero; Cicero speaks in his writing *Lucullus* (38, 119) of the "golden river of speech" of Aristotle. Aristotle's collected library went as an inheritance to his successor Theophrastus (Strabo 13, 1,54). The latter in turn bequeathed the books to Neleus, who brought them to his hometown Skepsis (Asia Minor) (Diogenes Laertius V, 52). Large parts of the books were acquired by Ptolemy II for the library in Alexandria. The remaining books were finally acquired by Apellicon of Teos and brought to Athens.

Upon the conquest of Athens in 86 BC, Sulla confiscated these books and brought them as war booty to Rome (Plutarch, Sulla 26). There the writings were collected by Andronicus of Rhodes and edited as the *Corpus Aristotelicum*. The works were arranged by subjects; the 14 books, which were *located after* the physical writings, were given the name *Metaphysics*, a bibliographic designation that only arose later.

Criticism of Plato's doctrine of ideas

Plato's statement that the *ideas exist independently of the perceptible things* arouses the criticism of Aristotle [Metaph. 1086B]:

> Without the general, it is in fact impossible to pursue a science, for distinguishing [the general from particular things] is the reason for the difficulties that arise with regard to ideas.

A large part of Book XIII of his *Metaphysics* is devoted by Aristotle to the examination of Plato's doctrine of ideas. Some points of criticism in abbreviated form are:

a) *What are the ideas good for anyway?* According to Aristotle's view, the ideas explain nothing; they are therefore of little help, if not even superfluous. If one can act well without having *seen* the idea of the good, why does one need these ideas [Nikom. Eth. 1097]?

b) *Problems with overlap:* The good and the beautiful are different from each other, since the good is always found in an action, but the beautiful also in the immovable. Mathematics, however, corresponds in the highest degree to the good and the beautiful [Metaph. 1078a]. It is unclear how this contradiction is to be overcome.
c) *How can one see ideas only through thinking?* When looking at beautiful things, one can only participate in the idea of beauty through sight; here, therefore, a sense perception is necessary. Plato argues here that the recollection of the soul makes the contemplation of ideas possible. This does not explain anything, because it only shifts the problem of perception to the soul.
d) *How does the realm of ideas behave?* Is there a one-to-one mapping between ideas and their manifestations? Is there therefore an idea for each thing and vice versa? The ideas are certainly not the cause if things change, as Heraclitus is convinced that all sense things are in a constant flux (πάντα ῥεῖ) [Metaphys. 1078b].

Zenon's Paradoxes

Zenon of Elea (480-430 BC) appears in Plato's dialogue *Parmenides* as such a skilled speaker that Aristotle attributes to him [Zenon] the invention of dialectics [Diog. Laert. IX, 25]. Aristotle discusses and debates four of Zenon's paradoxes of motion in detail in his *Physics* (239B). The three simpler ones are mentioned here: a) dichotomy, b) the arrow and c) Achilles.

a) Zenon claims that no body can move from *A* to *B*. For before it can reach *B,* it must first cover half the distance *AB,* that is |AB|/2. But before it can cover this distance, it must first cover |AB|/4, that is, half of it; the argument proceeds in a similar way. Since the distance *AB* can be divided an infinite number of times, the halving process can be carried out an infinite number of times. Since, according to Zenon, the sum of these intervals cannot be finite, motion is impossible.
b) As the arrow moves from *A* to *B,* there are an infinite number of points between *A* and *B* that the arrow must pass through. If we set the time equal to zero that the arrow needs to pass through one point, then the arrow is at rest in these points. According to Zenon, the existence of an infinite number of points in time at which the arrow is at rest prevents the arrow from moving.
c) Zenon claims here that the fast Achilles can never catch up to a slow turtle that has a certain lead. For in the time that Achilles runs the initial lead of the turtle, the turtle gains a new lead. This new lead Achilles must catch up to again and so on. Since there are an infinite number of points in time where the turtle has a lead, Achilles can never reach the turtle according to Zenon.

8.2 Mathematics with Aristotle

Aristotle laid the foundation for the following sciences with his writings: dialectic, analytics, metaphysics, logic, economics, politics, ethics, rhetoric, poetics, biology. His writings on physics impeded progress until Galileo; the geo-centric world view he propagated could only be overcome by Copernicus. A purely mathematical work of his is not handed down; however, he had a comprehensive understanding of mathematics. This is shown by the large number of remarks scattered throughout his works; they also represent a *valuable source* on the state of pre-Euclidean mathematics.

A typical quote from *Metaphysics* (1051a) shows the knowledge of Euclid I, 32 (angle sum in a triangle) and III, 31 (Thales circle):

> Theorems are only found in mathematics through active effort; for we only discover them through the process of analysis. If the [appropriate] analysis of a figure has already been carried out, then the theorems are immediately apparent; otherwise they are potentially hidden.
>
> Why is the sum of the angles of a triangle equal to two right angles? Because the angles around a point are equal to two right angles. If now the line running parallel to one side were drawn, then the situation would be immediately clear to everyone who has seen the first theorem.
>
> Why is an angle in a semicircle always a right angle? If the three lines are equal, namely the base line is twice as long and the perpendicular erected in the middle is once as long, then the matter is clear to everyone at first glance who has seen the first theorem.

A) Aristotle's proof of the angle sum was also adopted by Eudemus:
Proclus writes in his commentary on Euclid:

> Eudemus, the Peripatetic, attributes the discovery of the theorem that the interior angle sum of any triangle is two right angles to the Pythagoreans.

The proof uses the parallel axiom. Let ABC be any triangle with the parallel DE to BC through the point A. The angles $\measuredangle ABC$ or $\measuredangle BCE$ are congruent alternate angles at parallels, likewise $\measuredangle BAC$ or $\measuredangle ACD$. The angle $\measuredangle DCE$ is of course a straight angle; thus (Fig. 8.3)

$$\angle DCE = \angle DCA + \angle BCE + \angle CAE = 2R \Rightarrow \angle ABC + \angle BAC + \angle BCA = 2R$$

B) The proof to the Thales circle proceeds as follows: To be shown is: $\measuredangle\, BAC = R$ (Fig. 8.4).

The perpendicular AM decomposes the triangle—here assumed to be isosceles—into two isosceles triangles. The side BA is extended beyond A to D. The angles $\angle MAB$ and $\angle MBA$ are congruent because of $|AM| = |MB|$. In the same way, $\angle MAC = \angle MCA$ follows. For the angle sum at A one thus obtains:

$$\angle BAC = \angle BAM + \angle MAC = \angle ABM + \angle MCA$$

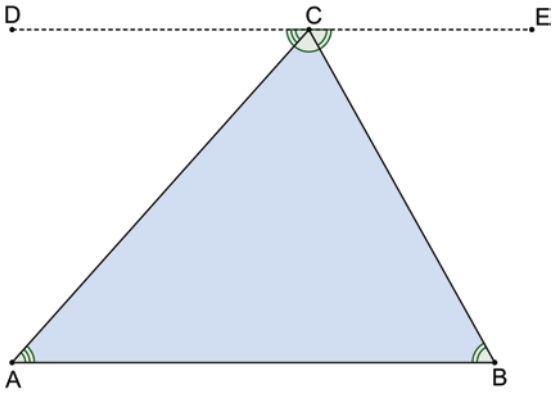

Fig. 8.3 Alternative of Aristotle to the angle sum

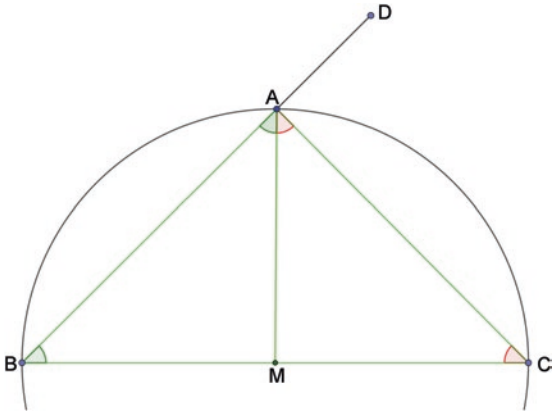

Fig. 8.4 Alternative of Aristotle to the Thales circle

The sum of the base angles is equal to the exterior angle $\measuredangle DAC$ [Euclid I, 32], thus

$$\measuredangle BAC = \measuredangle DAC$$

Since the angle $\measuredangle BAD$ is a straight angle, it follows that

$$\measuredangle BAC = \measuredangle DAC = R$$

According to Axiom [Euclid I, 10], all right angles are equal to each other.

Aristotle here assumes an equilateral triangle in a semicircle. This is not necessary for the proof, as his quotation on the same subject in Anal. Post. II [94] shows.

It is also characteristic that he repeatedly addresses sentences that are unsatisfactory from a philosophical point of view, such as the parallel axiom and the angle sum theorem. Surprisingly, a theorem is found in his writings that is missing in Euclid. This is the theorem about the geometrical locus, which will be much later attributed by Eutocius to the 150-year-later-born Apollonius (see below (9)).

Mathematical topics with Aristotle

Here is a selection of mathematical topics from Aristotle's writings in modern formulation:

1. In a circle, congruent chords enclose congruent circumference angles [Anal. Prior I, 24].
2. Base angles in an isosceles triangle are congruent [Anal. Prior I, 24].
3. When a line is cut by a parallel pair, the vertex angles are congruent [Anal. Prior II, 17].
4. The interior angle sum in a triangle is equal to two rights 2R [Anal. Post. I, 35] [Metaph. IX, 9].
5. The circumference angle in a semicircle is a right [Anal. Post. I, 1] [Metaph. IX, 9].
6. Determination of the mean proportional between two line segments [De anima II, 2] [Metaph. III, 2].
7. The sum of the exterior angles of a quadrilateral is equal to four right angles 4R [Anal. Post. II, 17].
8. The legs of an isosceles triangle are greater than the altitude drawn from the apex [*De incessu animalium* 9].
9. The geometrical locus of all points that have a given distance ratio $\frac{a}{b}$ between two given points is a circle [Meteor. III, 5].
10. If a straight line intersects a parallelogram, then the line divides the side and area of the parallelogram in the same ratio [Topik VIII, 3].
11. Proportions satisfy $a{:}b = c{:}d \Rightarrow a{:}c = b{:}d$ [Anal. Post. I, 5; II, 17] [Nikom. V, 3].
12. The ratio of the diagonal to the side of a square is incommensurable [Anal. Prior I, 24; I, 44].

Syllogism

The Greek word *Syllogismus* (Συλ-λογισμός) consists of the prefix *syl* (together) and the word *logismus* (calculation, consideration). Aristotle uses the term in the sense of *valid deductive conclusion.* He defines it several times in his writings:

> A syllogism is a *logos* in which, after certain assumptions [the premises], something different from these assumptions [the conclusion] necessarily results from the assumptions.

A syllogism is therefore a logically ordered sequence of sentences, some of which serve as premises and one as a conclusion. Well-known examples are

All men are mortal.	*No rectangle is a circle.*
Socrates was a man.	*All squares are rectangles.*
$\Rightarrow$ *Socrates is mortal.*	$\Rightarrow$ *No square is a circle.*

There are four types of statement forms (with their set-theoretic notation), as the table shows:

Type	Judgement	Notation
A	All S are P	$S \subseteq P, S \neq \emptyset$
E	No S are P	$S \cap P = \emptyset, S \neq \emptyset$
I	Some S are P	$S \cap P \neq \emptyset$
O	Some S are not P	$S \not\subseteq P$

A mnemonic rule for the letters of the a-, e-, i-, o-statements are the two words: *affirmo* (I affirm) and *nego* (I deny).

Statements about non-empty sets are always assumed, i.e., if all S are also P, then P also exists. Furthermore, there is a logical connection between general and specific statement, since the specific statement results from the general statement:

From SaP (*all S are P*) follows SiP *(some S that are P).*

From SeP (*no S is P*) follows SoP *(some S that are not P).*

Examples of statement rules are:

Premise 1 (Major premise): *All men (M) are mortal (P).*

Premise 2 (Minor premise): *All Greeks (S) are men (M).*

Conclusion (Conclusion): *Therefore, all Greeks (S) are mortal (P).*

The position of the predicates (here M-P, S-M, S-P) indicates the type of conclusion that is called *figure*. The three figures of the following table are from Aristotle; they were supplemented by a fourth figure in the Middle Ages:

	1. Figure	2. Figure	3. Figure
First premise	M—P	P—M	M—P
Second premise	S—M	S—M	M—S
Conclusion	S—P	S—P	S—P

These three figures can be written formally (with the syllogistic connection x) as

$$(M\,x\,P) \wedge (S\,x\,M) \Rightarrow (S\,x\,P) \tag{8.1}$$

$$(P\,x\,M) \wedge (S\,x\,M) \Rightarrow (S\,x\,P) \tag{8.2}$$

$$(M\,x\,P) \wedge (M\,x\,S) \Rightarrow (S\,x\,P) \tag{8.3}$$

Since x stands for one of the four syllogistic relations (a, e, i, o), there are theoretically a total of $3 \times 4 \times 4 \times 4 = 192$ modes; but only 24 of these are valid. All of these modes have been given mnemonic names since the Middle Ages. The four best-known modes are

$$(M\,a\,P) \wedge (S\,a\,M) \Rightarrow (S\,a\,P) \quad (B\underline{a}rb\underline{a}r\underline{a})$$

$$(P\,e\,M) \wedge (S\,a\,M) \Rightarrow (S\,e\,P) \quad (C\underline{e}l\underline{a}r\underline{e}nt)$$

$$(M\,a\,P) \wedge (M\,i\,S) \Rightarrow (S\,i\,P) \quad (D\underline{a}r\underline{i}i)$$

$$(M\,e\,P) \wedge (S\,i\,M) \Rightarrow (S\,o\,P) \quad (F\underline{e}r\underline{io})$$

The *e*-relation can be understood as the negation of the *i*-relation, and vice versa; similarly, the *a*-relation as the negation of the *o*-relation

$$S\,e\,P \Leftrightarrow \neg(S\,i\,P)$$

$$S\,a\,P \Leftrightarrow \neg(S\,o\,P)$$

A detailed description of syllogistics can be found in Chapter[4] (IV, 33) of the Aristoteles-Handbuch (ed. Rapp, Corcilius).

According to Prof. Marco Malink[5] (Chicago), Aristotle is considered the founder of formal logic. His writing *Organon* is said to have influenced the development of Western logic in an exemplary way. Such a presentation exceeds the scope of the book. However, the two most important principles of his logic (Metaph. IV, 3–8) should be mentioned:

- Principle of non-contradiction: It states that a sentence and its negation cannot both be true.
- Principle of the excluded third: This states that a sentence and its negation cannot both be false, but one of them is always true: $P \vee (\neg P)$.

This is related to Aristotle's view that no sentence can follow from its contradictory:

$$\neg(\neg P \Rightarrow P)$$

Furthermore, the principle of the contradiction proof also results. If you want to prove that $R \wedge S \Rightarrow T$, is true, you must show $(\neg T) \wedge S \Rightarrow (\neg R)$.

Finally, a quote by H. Flashar[6]:

> If you look at [Aristotle's] life as a whole, you stand in amazement before an achievement that is enormous for the conditions of his time as well as for today, and that manifests itself in a work that is so self-contained and homogeneous that you would never guess the difficult external conditions under which it came about.

[4] Malink M.: Syllogism, pp. 343–348, in the anthology: Aristoteles-Handbuch.

[5] Malink M.: Logic, pp. 480–484, in the anthology: Aristoteles-Handbuch.

[6] Flashar H.: Life (p. 5), in the anthology: Aristoteles-Handbuch.

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Alexandria

9

> *Are we not dead and only imagine ourselves to be alive, we Greeks, who plunged deeply into misfortune and only saw life in a dream? Or do we live indeed—while the true life passed away?* (Lament of the poet Palladas over the destruction of the Museion [from Anthologia Graeca, X, 82])

After the death of Alexander in 323 BC, the empire was divided among his generals Antigonus (Macedonia), Lysimachus (Asia Minor), Seleucus I (Mesopotamia and Syria), and Ptolemy I (Egypt and Palestine). Since Ptolemy's mother Arsinoe lived at the court of Philip of Macedonia, it is possible that he was related to Alexander by blood. In 305 BC he proclaimed himself king and took the name Ptolemy I Soter (= Savior), because as Alexander's bodyguard he had saved his life. The seizure of power after Alexander's death is vividly described in the Bible in the First Book of the Maccabees (1 Macc., 1,1).

> And it happened, after that Alexander son of Philip, the Macedonian, who came out of the land of Chettiim, had smitten Darius king of the Persians and Medes, that he reigned in his stead, the first over Greece, and made many wars, and won many strong holds, and slew the kings of the earth...And his servants bare rule every one in his place. And after his death they all put crowns upon themselves; so did their sons after them many years: and evils were multiplied in the earth.

The fight against Seleucus' successor Antiochus is described as follows (1 Macca., 1:16):

> When the kingdom was established before Antiochus, he thought to reign over Egypt that he might have the dominion of two realms. Wherefore he entered into Egypt with a great multitude, with chariots, and elephants, and horsemen, and a great navy, and made war against Ptolemee king of Egypt: but Ptolemee was afraid of him, and fled; and many were wounded to death. Thus they got the strong cities in the land of Egypt and he took the spoils thereof.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_9

Despite the Diadoch wars, Egypt began to develop into a cultural center in the Mediterranean under his rule. Not Athens, but Alexandria became—despite its location on the edge—with its economic wealth and the promotion of literature and science, the shining center of Hellenistic culture. Alexandria was sung by the poets and poets; Poet Herondas raves:

> For everything that there is in any way and that comes new in fashion, there is in Egypt: wealth, gymnasiums, power, pleasant climate, fame, cultural performances, scholars, gold, youths, the temple of the brother gods, a good king, the Museion, wine, and indeed everything good that one wishes.

There is a extensive description of Alexandria by Strabo, who lived in Alexandria for a long time; here is an excerpt (Geography XVII, 1, 8):

> The whole city contains convenient streets for riders and chariots, two of which—more than 100 feet wide—are the widest, which intersect at right angles. The city contains the most beautiful public temples and royal palaces, which make up a quarter or even a third of the whole area. For, as each king added new adornments to the public buildings out of a love of beautification, so each one built a palace next to the existing ones, so that now the poet's saying is applicable: *One thing is made from another*. All the buildings are connected and have access to the harbor, even those built outside (Fig. 9.1). Part of the royal buildings is also the Museion, which comprises a colonnade, a vestibule with seats and a large building that serves as a dining room for the scholars employed at the Museion. The association of these men manages the income together; at that time the priest appointed by the king was in charge of the Museion, now the head is elected by Caesar.

Ptolemy I himself had literary interests; he wrote a biography of Alexander the Great. His son Ptolemy II had two years (285–283) as an official co-regent of his father's government; with his death in 283 BC, he became sole ruler. His half-brother Ptolemy Crenaeus was deposed; he tried to gain influence through his marriage to their half-sister Arsinoe II, who was previously linked to the Thracian king Lysimachus. But Arsinoe II fled to Ptolemy II, who, after banishing his wife Arsinoe I, married her. He was therefore given the nickname Philadelphus (= sibling lover). Arsinoe II adopted her brother's three children, so that the

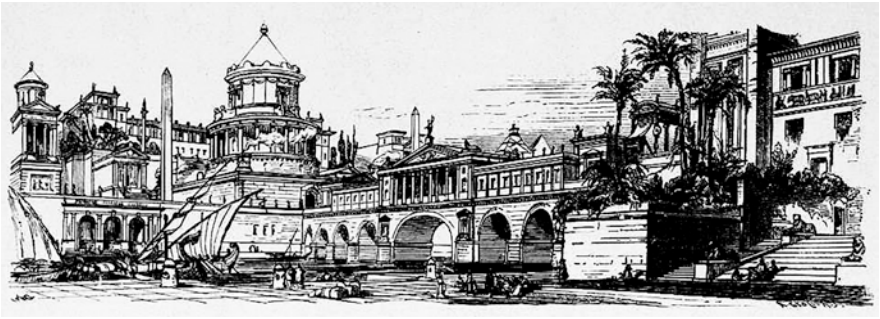

Fig. 9.1 Architectural drawing of Alexandria (Wikimedia Commons)

Fig. 9.2 Approval of the Museion by Ptolemy I, Göll 1876

oldest son Ptolemy III, later called Euergetes (= benefactor), could succeed him in government.

One reason for Alexandria's cultural ascent was the founding of the Museion (= seat of the Muses) and the library. Figure 9.2 shows Ptolemy I at the building permit for the Museion.

The siblings' couple ensured economic growth through a rigorously planned economy that declared the whole country the Pharaoh's personal property. A well-educated, strictly hierarchical group of officials not only supervised grain cultivation, but also fruit cultivation and animal husbandry. By establishing monopolies on the olive trade, papyrus production and banking, the ruling house generated great revenue and could make itself popular through acts of charity. The double harbor and the construction of the famous lighthouse (*Pharos,* Greek. φάρος) (Fig. 9.3) were the symbol of this economic success, which made Alexandria the trading center of the entire Mediterranean region. The name of the builder of the *Pharos* is known from an inscription found later:

ΣΩΣΤΡΑΤΟΣ ΔΕΞΙΦΑΝΟΥ ΚΝΙΔΟΣ
ΤΟΙΣ ΘΕΟΙΣ ΣΩΤΗΡΣΙΝ
ΥΠΕΡ ΤΩΝ ΠΛΩΙΖΟΜΕΝΩΝ
Sostratus of Knidos, son of Dexiphanus, To the saving Gods [dedicated], For the seafarers [erected].

Fig. 9.3 Artistic image of the Pharos, GetArchive LLC

It is part of an epigram by the poet Posidippus of Pella, who lived during the reign of Ptolemy II:

> For the salvation of the Greeks this lighthouse was erected as a watchman, O Lord Proteus, by Sostratos son of Dexiphanos of Cnidus. Because in Egypt there are no vantage points on a mountain, as on the islands, the breakwaters of the harbor where ships dock are low. This tower, built like a vertical pillar, seems to hold the sky high during the day. At night the sailor at sea sees the great fire glowing at its top.

Ammianus Marcellinus reports a tsunami in 365 AD destroyed the lighthouse. By 1183, the lighthouse was restored, as can be seen from the travel report "Diary of a Pilgrim to Mecca" by ibn Dchubair[1] from which one can infer:

> Among the greatest of the wonders that we ourselves have seen is the lighthouse that Allah has founded with the hands of those to whom he has entrusted this front work […] as a guide for travelers. […] The interior of the lighthouse, because of its spaciousness, its steps, its halls, offers an overwhelming sight […] On the top there is a kind of mosque, a blessed place […] On Thursday, April 1, we climbed the lighthouse and performed our prayers in this praised mosque. There we saw such wonders of architecture that they cannot be described in words.

[1] Ibn Dchubair: Diary of a Pilgrim to Mecca, p. 24, Library of Arab Storytellers, Goldmann 1988.

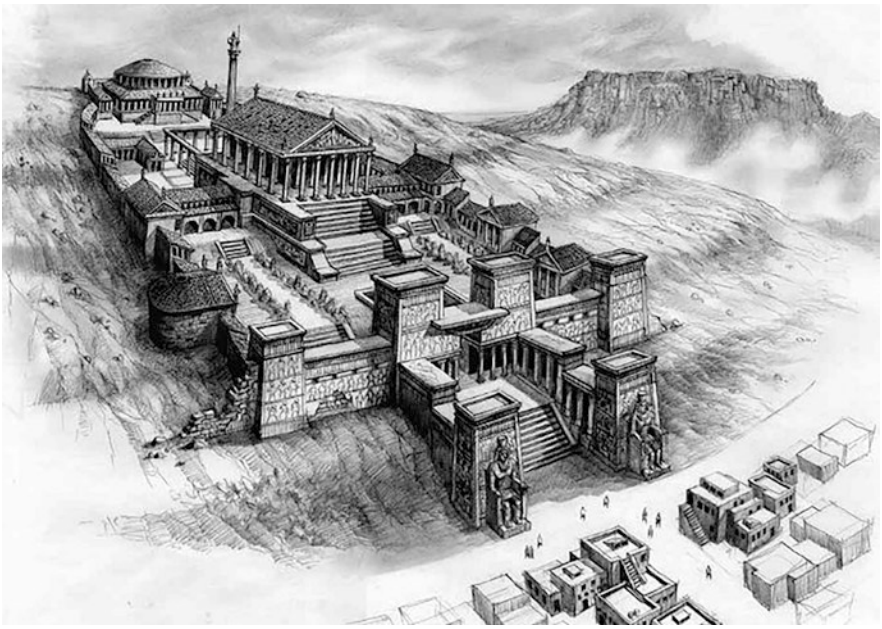

Fig. 9.4 Building complex of the Serapeion (Wikimedia Commons)

The taxes that the traders had to pay from all the countries were regulated in such a way that a good part of the profit remained with the trade. The temples and monasteries were dealt with less gently; the temples were forced to hand over their income from donations to the royal house. Arsinoe II founded a cult on a new god, who was strangely a personal union of the god *O-Siris* and the bull god *Apis*; his name was *Serapis,* composed of "Siris" and "Apis". The name was appropriately chosen, as there was already a god with the same name Sarapis/Serapis, who was represented with a fruit basket on his head (as a symbol of fertile Egypt). The newly founded temple complex, *Serapeion* called, was built on an artificial hill and lavishly equipped (Fig. 9.4). It was not made up of a single building, but was a huge number of buildings, all grouped around a central building, surrounded by columns of enormous size and graceful proportions.

The luster of the Alexandrian court and the royal patronage of the sciences attracted the most famous scientists to Alexandria. These were, on the one hand, the poet Philitas of Cos, the grammarian Zenodot and the physicist Straton. After the latter had worked as a tutor of the king's son, he returned to Athens in 288 BC as a librarian of the Lykeion.

Straton is not to be confused with the geographer Strabon, who in the year 25 BC undertook a Nile voyage with the emperor Aelius Gallus and visited Alexandria on this occasion:

Fig. 9.5 City map Alexandria (a) (Wikimedia Commons), Surrounding map (b), colored by author

> The *Museion* belongs to the area of the royal palaces; it contains a public hall for walking and study rooms for sitting, as well as a large building with a dining room for the students. The community of men manages their property together; the head is a priest, who is appointed by the king, now by the Roman emperor. […] Beyond the canal, however, lies the *Serapeion*, as well as other sacred buildings, which are now almost abandoned because of the new buildings in Nikopolis.

Figure 9.5a shows the city map of Alexandria, Fig. 9.5b the surrounding map with the towns of Nikopolis and Kanopus; the latter town was later called Abu Kir; this is the site of the sea battle in which Admiral Nelson's ships destroyed Napoleon's French fleet.

For the operation of the library, the cataloging of the book stocks is of great importance; it was invented by the philologist and poet Callimachus of Cyrene (320/303–240 BC). The result of his work was a 120-roll author index, which listed a short biography and a work index for each Greek author (called pinakes) and thus created the first scientific library catalog in the world. His saying is known:

> A big book is a big evil (μέγα βιβλίον μέγα κακόν)

which certainly referred to the difficulty in cataloging. Several preserved poems on Ptolemy II and his second wife Arsinoë II. prove Kallimachos to be the court poet of this royal couple. The numerous papyri found with his writings and his frequent quotation suggest that Kallimachos was one of the most widely read authors of his time. Alone in the Greek Anthology there are 63 epigrams by him. His poetry had a great influence on the Roman authors Catull, as well as Properz and Ovid. Among Kallimachos' pupils were, in addition to Apollonios of Rhodes, Eratosthenes of Kyrene and Aristophanes of Byzantium.

His most famous poem (last in his work *Aitia* = Origins poem) tells the legend of Berenike's hair. Berenike II, the wife of Ptolemy III, had made a vow to the goddess Aphrodite that she would sacrifice her hair if her husband returned unharmed from the Syrian campaign (241 BC). The hair was kept in the temple

of Arsinoe Zepherites and was suddenly gone through the work of the goddess. Days later, the astronomer *Konon* of Samos reported that he had seen the hair as a new constellation in the sky. The Hair of Berenike (Latin *coma berenice*) is still a constellation of the northern sky. For centuries, the poem by Kallimachos was only known in the Latin translation by Catullus from 66 AD until the original text was rediscovered in 1929 on a papyrus. Here is an excerpt[2] from Catullus' version; Catullus lets the hair speak:

> Not only *Ariadne* shone in that golden garland,
> Which once adorned her forehead, that next to her light might shine,
> I, the adornment of her blonde head, consecrated to the gods.
> Still wet with tears that led me to the temple,
> I was placed among the old as the youngest constellation,
> Where I, the light of the *Virgo* and the grim *Leo* touching,
> With the son of Lycaon, with *Callisto,* united,
> I move towards the evening as the leader of the sluggish *Bootes,*
> Which sinks down to the ocean in a tardy run.

In addition to *Virgo* and *Leo*, the following star constellations are mentioned here: Ariadne is the *Little Crown,* Callisto the *Great Bear,* Bootes the *Bear Keeper.*

It is probably no exaggeration to say that a large part of the surviving Greek literature has come to us either directly or indirectly from Alexandria. All writings of Greek poetry, especially the dramas and comedies, were edited there. The manuscripts of the philosopher Plato received special attention. The works of Homer *Iliad* and *Odyssey* were divided into 24 songs each; curiously, the story of the Trojan Horse cannot be found in the Iliad; it was included in the 8th song of the Odyssey [line 492 ff.].

Alexandria was a gathering point for Jews, some of whom even migrated from Syria – also belonging to the Ptolemaic Empire – and soon formed their own district. Many of the Jews were hellenized and adopted the Greek language; Jewish scholars translated parts of the Old Testament, particularly the first five books of Moses *(Pentateuch)* into Greek (123 BC). The apocryphal book of the Bible, the so-called *Wisdom of Solomon,* was also written by a Jewish author in Alexandria at this time. The verse Wisdom XI, 20–21 clearly shows Pythagorean thought:

> But thou [Lord] hast ordered all things in measure and number and weight.

The verse Wisdom XIII, 1 reveals knowledge of early Greek philosophy, which is however rejected as pagan:

> Surely vain are all men by nature, who are ignorant of God, and could not out of the good things that are seen know him that is: neither by considering the works did they acknowledge the workmaster; but deemed either *fire*, or *wind*, or the swift *air*, or the circle of the stars, or the violent *water*, or the *lights of heaven*, to be the gods which govern the world.

[2] Catulli carmina 66, 59–68, in Balss H. (ed.): Antike Astronomie, Tusculum München 1949.

This alludes to the primordial substance *water* (in relation to Thales) or the primordial substance *air* (in the case of Anaximander); in the case of Empedocles, the *immutable Being* is a mixture of the four elements: fire, air, earth, water. The "lights of heaven" point to astrology. The Bible translation led to internal disputes. The Egyptians did not like that the biblical account of the Exodus. Moses' splitting of the Red Sea was celebrated as a heroic victory of the Jews over Egypt. In *Exodus* (15, 1), (15, 21) it says about this:

> Let us sing to the Lord, for he has done a glorious deed; *horse and rider he has cast into the sea*.

Mathematics and the natural sciences also flourished in Alexandria. Euclid himself worked during the reign of Ptolemy I; but his works were only later edited by Theon of Alexandria and brought into the form and sequence that we know. Ktesibios and his later successor Heron of Alexandria also caused a sensation in the temples of Alexandria with their spectacular devices and machines (such as automatic door openers and holy water dispensers).

9.1 The Library

The Library of Alexandria (Fig. 9.6) was indeed the most famous of its kind, but by no means the only one. Other libraries existed in Athens, Pergamon, Rhodes, or Smyrna (today Izmir/Turkey). According to tradition, the library already contained about 200,000 papyrus rolls during the lifetime of Ptolemy II. Allegedly, every ship that anchored in Alexandria had to deliver copies of the scrolls on board which were not already in the library. When Demetrius of Phaleron, the former city administrator of Athens, had to flee from Athens, he was given asylum by Ptolemy II. In 284, he appointed him the first librarian and, following the example of the Lyceum, had him set up the library. *Aristeas* Judeos reports that Demetrius had such a large budget that he could buy all the books if possible, otherwise have them copied. Athenaeus of Naucratis tells us that books by Aristotle and his successor Theophrastus were acquired from the Lyceum by unfair methods. At the time of Caesar, according to the Byzantine scholar Johannes Tzetzes, there were 500,000 scrolls in the library.

The Papyrus P. Oxy. 1241 handed down some names of librarians, including some well-known persons

- Zenodotos of Ephesos
- [Kallimachos of Kyrene]
- Apollonios of Rhodes
- Eratosthenes of Kyrene
- Aristophanes of Byzantium
- Apollonios Eidographos
- Aristarchos of Samothrace

Fig. 9.6 The interior of Alexandria's library, AKG5700772 Copyright/Heritage Images/Fine Art Images/akg-images

The name of the poet Kallimachos is not mentioned; since he wrote the pinakes, he was certainly busy at the library. So his name was added to the list here. Further names are not handed down, since the library obviously became insignificant after 145 BC.

After the Punic Wars (264–146 BC), Carthage was destroyed and North Africa came under Roman rule. Alexandria was initially unaffected by Roman politics and was able to trade unhindered. The city suffered a loss of reputation when Ptolemaios VIII had some unpopular Greek scholars killed in a purge in 145 BC. Once even a soldier from a unit of spear-carriers(!) was elected the head of the library.

The influence of Rome became apparent when Alexandria was drawn into the civil war between Julius Caesar and Pompeius (48 BC). After Pompeius lost the Battle of Pharsalos to Caesar, he fled to Alexandria to seek refuge. In order to do a favor to Rome, he was killed there on the orders of the co-regent Ptolemaios XIII. Caesar showed himself to be "outraged" and launched an attack on Alexandria with his fleet. When the Alexandrian fleet tried to block the Roman ships in the harbor, Caesar set the ships on fire. Since the Alexandrians had also barricaded the land routes with large wooden obstacles, a great fire broke out (Fig. 9.7); in the

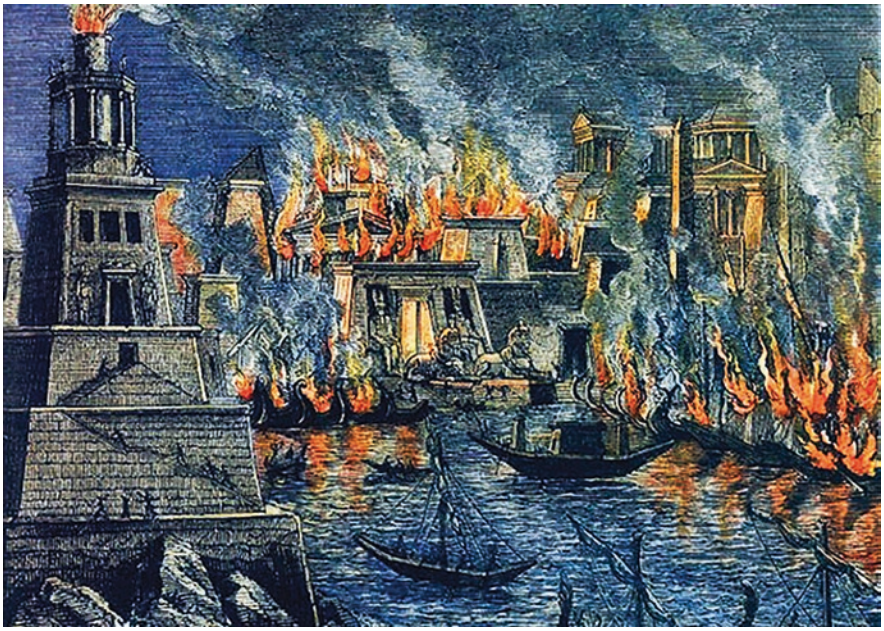

Fig. 9.7 The fire of Alexandria (Wikimedia Commons)

process, all the scrolls stored in the harbor were burned. As Caesar reports in his work *De bello Alexandrino*, a total of 110 ships[3] were destroyed.

After Caesar's murder in 44 BC, Marcus Antonius married Cleopatra and fled with her to Alexandria. According to Plutarch, Marcus Antonius compensated for the loss of books by Caesar by giving Cleopatra the books from the library of Pergamon. From Alexandria, he led a 13-year war against Rome until he was defeated by Octavian at the Battle of Actium in 31 BC. With the suicide of Cleopatra, the dynasty of the Ptolemies came to an end. Egypt became a Roman province and was obliged to serve as the breadbasket of the Roman Empire.

There were two major Jewish revolts against the Roman occupation, the Jewish War and the Bar-Kochba Revolt; Alexandria was again destroyed in 115 AD. Emperor Hadrian had the city rebuilt, which he showed great interest in as a scholar. Hadrian visited the province of Alexandria for 8 months; however, his poems to the city have been lost. The emperor's epigram to Pompey, whose tomb he visited in Pelusium, has been preserved: "He who was so rich in temples is as poor and small in his grave" (Greek Anthology IX, 402).

To quell an alleged uprising, Emperor Caracalla came to Alexandria at the turn of 215/16 AD and perpetrated a terrible massacre. Under Emperor Aurelius, the

[3] Caesar G. J., Jahn C.(ed.): Bellum Alexandrinum, p. 18, in the anthology: Kriege in Alexandrien, Afrika und Spanien.

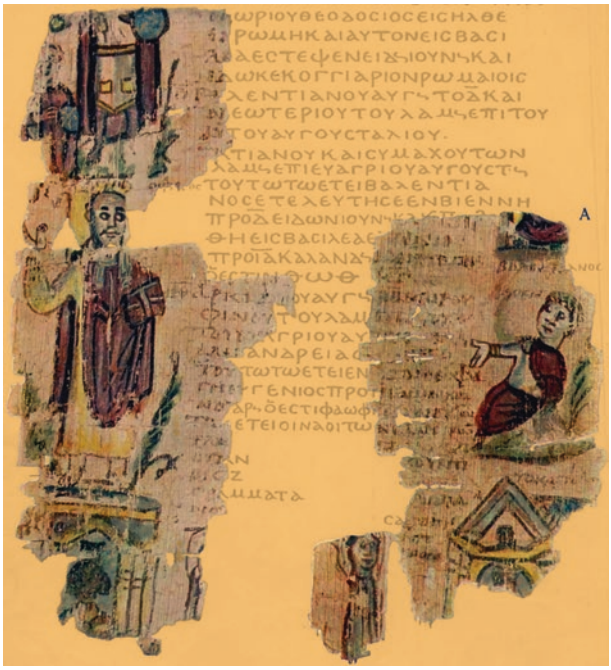

Fig. 9.8 Triumph of Theophilus (Wikimedia Commons)

province of Egypt was briefly lost to the Kingdom of Palmyra in 269–270. After its recapture, it became part of the Eastern Roman Empire's sphere of influence. Emperor Constantine the Great issued the Edict of Milan in 313 AD, Christians were granted equality in religious matters. Finally, under Eastern Roman Emperor Theodosius I, paganism was outlawed. In 365 AD, a powerful earthquake (as mentioned) destroyed large parts of the coastal strip from Alexandria to Heraklion.

At the urging of the fanatic Bishop of Alexandria Theophilus, Theodosius had the Serapeion destroyed in 391 AD. The Golenischev Papyrus shows Theophilus, triumphantly standing on the ruins of the destroyed Serapeion with the Bible in his hand (Fig. 9.8). The destruction of Alexandria's libraries resulted in a tremendous, irretrievable loss of ancient literature, the extent of which cannot be estimated in any way. *After the death of Hypatia, no philosopher in Alexandria troubled himself*, writes Gibbon, the famous author of the book *The Decline and Fall of the Roman Empire.* But the religious wars were not over yet. In 412 Cyril, the uncle of Theophilus, became Patriarch of Alexandria. He was a fanatic zealot and launched a pogrom against the numerous Jews there. He also eliminated his competitor Nestor, the Patriarch of Constantinople, by declaring him a heretic at the Council of Ephesus (431). In gratitude for his services, he was later declared a saint.

Ammianus Marcellinus (Res Gestae XXII, 16, 17) writes in 363 AD about the state of Alexandria (cited after[4]):

> There are also temples here with high gables. The Serapeum, which is to be described, is of particular importance. Its splendor is formed by wide colonnades, lively statues and many other works of art to such an extent that the world knows nothing more magnificent, with the exception of the Capitol, with which the venerable Rome rises to eternity. In it [Serapeion] were two invaluable libraries. As the old testimonies agree with reliability, 70 000 volumes, which the Ptolemaic kings had brought together with tireless care, were burned in the Alexandrian War of the dictator Caesar, when the city was plundered.

Later, he reports on the scientific atmosphere of the Late Period:

> Although, in addition to those I have mentioned, many flourished here in ancient times, the various sciences nevertheless do not even now fall silent in this city. For the teachers of these disciplines live somewhat, and with the rod of the mathematician one brings to light everything that is still hidden in the darkness; the music of the Alexandrians is not yet entirely extinguished, the harmony is not yet entirely mute, the consideration of the motion of the universe and of the stars is still kept warm with some, few though they be, and there are others who are skilled in numbers; in addition, few are acquainted with the knowledge that points out the ways of destiny. The studies of medicine, however […] flourish from day to day to such an extent that, for a doctor […] it is sufficient, instead of any proof, to say that he was trained in Alexandria.

It is possible that parts of the library have been preserved, since the Roman Aphthonius of Antioch, who visited Alexandria after 391 AD, still found intact buildings

> … on the inside of the colonnade there were rooms, some of which served as bookstores and were open to those who devoted themselves to scholarship. It was these study rooms that had made the city first in philosophy. Some other rooms were intended for the worship of the old gods.

In 619 AD, Alexandria was first conquered by the Persians under their general Chosrau. The troops of Byzantium were able to recapture the city in 628 under Emperor Heraclius, but lost it again to the troops of the Caliph Umar in 641 AD. The Muslim conquest was insignificant in terms of the library's book collection; Umar is said to have said according to Bar Hebraeus (Chronicum Syriacum):

> If these books are in agreement with the Quran, we have no need of them, and if these are opposed to the Quran, destroy them.

In the year 646 AD the Prophet's Muslim troops defeated Byzantium and all of Egypt finally fell under Islamic rule.

[4] Sartorius J. (ed.): Alexandria Fata Morgana, Wissenschaft. Buchgesellschaft o. J., pp. 235 ff.

Fig. 9.9 Underwater archaeology off the coast of Heraklion, Christoph Gerigk. (© Franck Goddio/Hilti Foundation)

Since a papyrus roll with Aristotle's treatise on the *Constitution of Athens* was found in the village of Oyxrhynchos (today Al Banasa) in 1890, systematic excavations have been carried out here. The most spectacular find was the discovery of the so-called Thomas Gospel. Since fragments of works by Pindar, Sappho, Sophocles and Euripides have been found, some authors have assumed that an outstation of the Alexandrian library took place here, since papyri are better preserved in the dry sand than in the wet subsoil of the Nile Delta. But the great distance to Alexandria speaks against this assumption.

The severe earthquake of 1330 that finally destroyed the lighthouse also caused large parts of the coast between Alexandria and Heraklion to collapse into the sea. In the years 1994–1998, a number of relics, statues and building remains were discovered in the harbor basin of Alexandria by the team of French underwater archaeologists Jean-Yves Empereur, some of which were recovered. Figure 9.9 shows the team of Franck Goddio diving off the coast of Heraklion. Since 2009, UNESCO has been planning to represent the artifacts remaining in the sea in an underwater museum; no implementation is in sight.

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Euclid of Alexandria

10

> *Geometry is here in its first elements, as it is in Euclid, and how we can let a beginner start. But then it is the perfect preparation, indeed the introduction to philosophy.* (J. W. Goethe on 28.02.1809 in conversation with the educator J.D. Falk)

Euclid (Fig. 10.1), actually called Euclides (Ἐυκλείδης), was probably born in Alexandria around 360 BC. Since Book XII and parts of Book V of his *Elements* originate from Eudoxus, many authors believe Euclid having studied in Athens at the Academy; this is by no means secure. He was appointed to Alexandria by Ptolemy I Soter (ruled 305–285 BC). Proclus reports about him:

> Not much younger than those [students of Plato] is Euclid, who put together the Elements, gathered much of what was found by Eudoxus, completed much of what was begun by Theaetetus, by bringing to irrefutable proof that which had been carelessly proved by his predecessors. He was born in the time of the first Ptolemy; Archimedes also mentions him in [his] first book.

The famous anecdote follows in Proclus's report:

> It is also said that Ptolemy Soter once asked him whether there might not be an easier way to geometry—other than that of the elements—to which he replied: *There is no royal road to geometry*. He is younger than Plato's pupils, but older than Eratosthenes and Archimedes, both of whom were contemporaries, as Eratosthenes reports.

Figure 10.2 shows Euclid's reception by King Ptolemy I Soter.

Until about 1950, the prevailing opinion was that Archimedes lived after Euclid. It was also known that Archimedes always spoke *of the elements* in relation to previous theorems, but never mentioned Euclid directly, with the exception of one passage in Theorem 2 of Chapter 1 in the treatise *On the Sphere and the Cylinder.* This aroused the interest of the Dane J. *Hjelmslev;* He quickly

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_10

Fig. 10.1 Euclid (Wikimedia Commons)

discovered that reference was being made here to the wrong Euclid theorem (I, 2), rather than to the approximately correct Euclid theorem (I, 3). Therefore, in an article[1] he expressed the suspicion that this must be an interpolation by someone else. He wrote (in German):

> But the reference is quite naive in any case and must have been inserted by a non-expert copyist.

His Dutch colleague E.J. *Dijksterhuis*[2] agreed with him, why should Archimedes cite the name Euclid in such a simple matter, when he never did so in any of his other writings? However, the insertion must have been made at an earlier time, since Proclus already speaks of the Euclid quotation from Archimedes. Since Alexandria was founded in 332 BC, one can set Euclid's work in a period of about

[1] Hjelmslev J.: On Archimedes' Theory of Magnitudes, Kogelike Dansk Videnskabernes Selskabs Skrifter, Matematisk-fysiske Meddelelser, 25, 15 (1950), p. 7.

[2] Dijksterhuis E.J.: Archimedes, Eijnar Munksgaard 1956, p. 150.

Fig. 10.2 Euclid's audience with Ptolemy I, Göll 1876

320 to 260 after about 10 years of construction. Since Archimedes was born about 287 BC, he may have been a contemporary of Euclid in Alexandria. Ptolemy Soter ruled from 323 and was king from 304 to 285.

The *Elements* are particularly praised by Proclus (in the translation by J.L. Heiberg[3]):

> But one should admire him especially in connection with the elements of geometry, because of their order and the selection of theorems and problems prepared for the elements. For he did not take everything that he could say, but only what could serve as an introduction. He also included reasonings of all sorts, both proofs founded on causes and proofs based on signs, so but all of them impeccable, exact, and appropriate to science. Besides these the book contains all the dialectical methods: the method of division for finding kinds, definitions for making statements of essential properties, demonstrations for proceeding from premises to conclusions, and analysis for passing in the reverse direction from conclusions to principles. The various forms of conversion, both the simple and the more complex, can be accurately learned in this treatise.

[3] Heiberg J.L.: Litterärgeschichtliche Studien über Euklid, Leipzig 1870.

P. Fraser[4] could show that the first mentioned Proklos-passage is not linguistically clear. It could also be interpreted as: *Euklid lived under the first Ptolemaios, because Archimedes, who was still his contemporary, quoted him in his first book.* But since *sphere and cylinder* was certainly not the first book of Archimedes, the previously interpretation seems appropriate.

Pappos tells at the beginning of book VII, that Apollonius *spent a long time with the pupils of Euklid in Alexandria.* This statement is difficult to reconsile with the usual life data of Apollonius (ca. 265–190 BC). He praises him as a good pedagogue:

> He [Euklid] was of the mildest disposition and, as befits, well-disposed to everyone, who, and were it but little, the mathematical disciplines could promote, in no way malicious to others, but in the highest degree considerate.

Whether Euklid has worked at the Museion or elsewhere is not known; at any rate, it is certain that he was not the librarian of the famous library, as is often read. Arabic sources report that the elements—allegedly written by an Apollonius—were scattered and then, on behalf of the king of Euklid, collected and systematically processed.

B.L. van der Waerden writes that Euclid was not a great mathematician; he only attributes didactic skills to him. Euclid certainly had the recognition of his countrymen. Proclus supplements in his aforementioned report on the proof of the Pythagorean theorem:

> I admire those who first investigated the truth of this theorem, but one must highly esteem the author of the Elements: He not only confirmed this theorem by the most convincing proof, but also in the sixth book [VI, 31] the even more comprehensive theorem by irrefutable scientific proofs [similarity].

T. Heath [I, 75] writes that the Elements do not contain a note on originality. However, Euclid made great changes to the arrangement of whole books, completely redistributed auxiliary theorems, developed new proofs, so that the understanding of the content was facilitated in the new arrangement. He did not allow changes or new insights that exceeded his knowledge. Furthermore, Euclid showed great respect for tradition by adopting some theorems that were useless or outdated.

B.A. Rosenfeld[5] explains the elements as a revision of older Greek writings. In his opinion, Books I–IV and XI are by Hippocrates of Chios, Books VII–IX by the older Pythagoreans, the theory of proportions (Book V–VI) and the method of exhaustion (Book XII) by Eudoxus, and finally the quadratic irrationalities (Book X) and the regular polyhedra (Book XIII) by Theaetetus.

It must also be mentioned that Euclid of Alexandria was confused for centuries with the Euclid of Megara, who lived about 100 years earlier. J. Itard[6], who

[4] Fraser P.M.: Ptolemaic Alexandria I, pp. 386–388.

[5] Rosenfeld B.A.: A History of Non-Euclidean Geometry, Springer 1988, p. 35.

[6] Itard J.: Les livres arithmétiques, Hermann 1961, p. II.

published the following three theses in an edition of the *Elements*, falls into a similar groove:

- Euclid was a single person who put together all the works that are attributed to him today.
- Euclid was a single person who was the head of a school whose students also published under his name after his death.
- Euclid was a group of Alexandrian mathematicians who published under the name Euclid of Megara.

Itard himself considers hypothesis b) to be the most probable.

At the beginning of Greek mathematics, there was a stage at which people believed that "everything" could be proven without having to rely on arbitrarily selected axioms. It was Aristotle who recognized that every knowledge system based on proofs must ultimately rest on unproven premises. He made the demand [Anal. Post. 72B29] that these first unproven premises must be more *intelligible* and *simpler* than the theorems derived from them or proven with their help. It was Euclid who followed Aristotle's demand and made the significant attempt to foundation geometry on axioms or postulates.

The first five of the 23 *definitions* by Euclid[7] are as follows:

1. A point is that which has no parts.
2. A line is a length without breadth.
3. The ends of a line are points.
4. A straight line (line segment) is one which lies evenly with the points on it.
5. A surface is that which has only length and breadth.

L. Russo[8] is of the opinion that these definitions originate from Heron of Alexandria. Comments by Sextus according to his opinion indicate that he is referring not to the *Elements,* but to the work *Definitions of Terms from Geometry,* which is attributed to Heron.

The definitions (1) to (5) appear naive today. Euclid still distinguished between postulates and axioms. The first four postulates are as follows:

It is required:

1. That one can draw a straight line from each point to every point.
2. That one can produce a limited, straight line continuously and straight.
3. That, with each center point and distance, one can draw a circle.
4. That all right angles are equal to one another.

[7] Euclid, Thaer C. (ed.): The Elements, Oswald's Classics Volume 235, Harri Deutsch 1997[3].

[8] Russo L.: The Forgotten Revolution, Springer 2005, pp. 369 ff.

The fifth postulate is the famous parallel axiom:

5. If a straight line intersects two straight lines, and if the interior angles on the same side of the straight line that is being intersected are less than two right angles, then the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles.

But it took more than two millennia for D. Hilbert[9] to present a first axiomatic system of geometry in the modern sense (1901). Hilbert had a predecessor in M. Pasch, who had already presented a first arrangement axiom in 1882. How far the formalism has progressed is shown by Hilbert's[10] famous saying, which he made spontaneously in a Berlin waiting room:

> You must be able to say "tables, chairs, beer mugs" at any time instead of "points, lines, planes".

The works

The *Elements* (Στοιχεῖα = Initial foundations), written around 325 BC, contained 13 books, which were later expanded to 15 books. Proclus (p. 214) writes about Euclid's other writings:

> He is also the author of a large number of other mathematical writings, all distinguished by their admirable accuracy and scientific speculation.

The following works by Euclid have been preserved:

- Data
- On the decomposition of figures
- Optica (Perspective)
- Catoptics (Reflections, optical illusions)
- Sectio Canonis (Music theory)
- Phainomena (Geography, astronomy)

The following works have not been handed down:

- Pseudaria (Fallacies)
- Conics
- Porismata (Fundamental theorems between theory and construction)

As one can see, Euclid delivered works on all of the disciplines that Plato demanded.

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[9] Hilbert D.: Die Grundlagen der Geometrie, Teubner 1968[13].

[10] Hilbert D.: Gesammelte Abhandlungen Band III, Springer 1970, p. 403.

10.1 From Book I of the Elements

Euclid I, 2

From a given point A, to draw astraight line equal to a given straight line BC.

Construction (Fig. 10.3):

Euclid gives the following construction: If BC is the given distance and $A \notin BC$ is the given point, then the equilateral triangle ΔABD is erected upon AB. DB is produced beyond B by |BC|; endpoint is E. Now DA is also produced by |BC|; endpoint is F. |AF| is now the desired distance.

Proof: The construction of the equilateral triangle ΔABD provides $|DA| = |DB|$. The circle with center B through C intersects the line BD at the point E. Thus $|BC| = |BE|$. The circle around D through E intersects the line DA at the point F. Thus $|DF| = |DE|$. Subtraction of the triangle side shows $|AF| = |BE| = |BC|$. Thus, at point A, the line segment AF congruent to BC is drawn.

Remark: The construction using an equilateral triangle appears to be complicated here, but it is necessary because Euclid does not allow for the transfer of lines using a fixed circle or marked ruler.

Euclid I, 5

In any isosceles triangle if the equal sides be produced, the external angles at the base are equal, and the internal angles at the base are also equal.

Given is the isosceles triangle ABC with the congruent sides $|AC| = |BC|$. These sides are produced by the congruent line segments $|AD| = |BE|$ (Fig. 10.4).

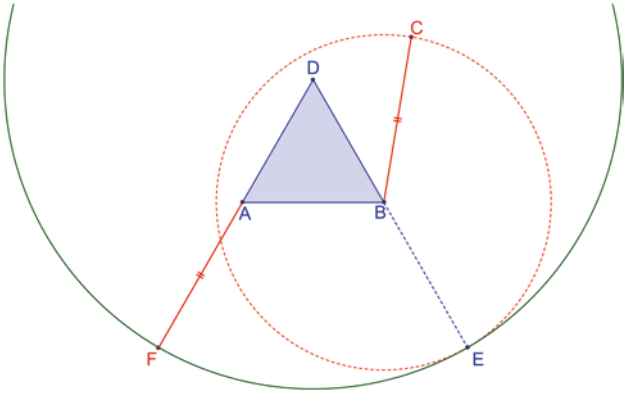


Fig. 10.3 Figure for Euclid I, 2

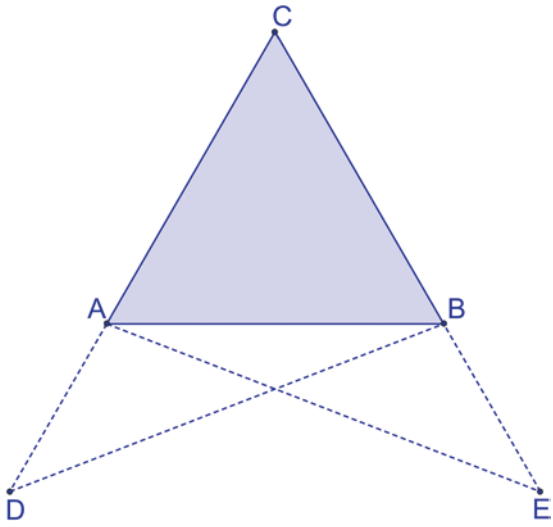


Fig. 10.4 Figure for Euclid I, 5

1. The triangles CDB and CAE are congruent according to the SAS theorem, since they correspond in two sides $|CD| = |CA| + |AD|$ resp. $|CE| = |CB| + |BE|$ and the common angle $\measuredangle ACB$. Thus, the angles $\measuredangle CAE$ and $\measuredangle DBC$ are also congruent.
2. The triangles $\triangle ADB$ and $\triangle AEB$ are congruent, since they arise from the congruent triangles $\triangle CDB$ resp. $\triangle CAE$ by removing the triangle $\triangle ABC$ contained in both. Thus, the angles $\measuredangle EAB$ and $\measuredangle DBA$ are also congruent.
3. Since the angles $\measuredangle CAE$ and $\measuredangle CBD$ and their partial angles $\measuredangle BAE$ resp. $\triangle ABD$ are congruent, the remaining angles $\measuredangle CAB$ and $\measuredangle ABC$ are also congruent.

Aristotle considers a twofold symmetry of the angles: Both the angles between the legs A, B of the triangle and the circular arc, resp. the angles between the chord and the arc, should be symmetrical. Subtraction of the latter angles from the former yields the congruence of the inner angles (Fig. 10.5). This proof attempt could go back to Thales.

Euclid I, 9

To bisect a given rectilinear angle.

Construction: Given is the (acute) angle BAC. Choose an arbitrary point D on AB. The circle with center A through D intersects AC at E, so that $|AD| = |AE|$. Construct an equilateral triangle $\triangle DEF$ upon the line segment DE. The line AF is then the desired bisector of the angle $\measuredangle BAC$.

Proof (Fig. 10.6): By construction, $|AD| = |AE|$ and $|DF| = |EF|$. The triangles $\triangle ADF$ and $\triangle AFE$ are congruent according to [Euclid I, 8], since they coincide in two sides and the common side AF. Thus, the corresponding angles are congruent and AF is the angle bisector, since

$$\measuredangle BAF = \measuredangle ACF$$

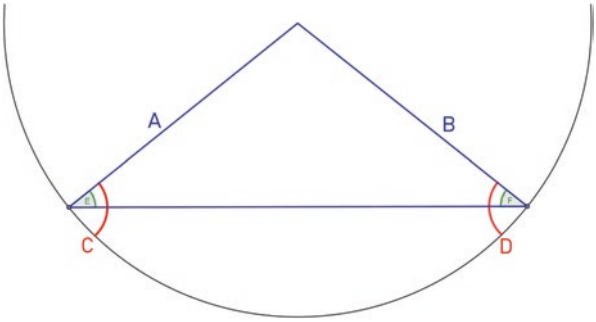
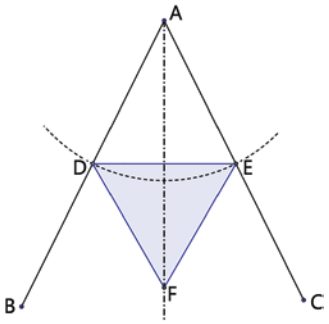


Fig. 10.5 Alternative by Aristotle

Fig. 10.6 Figure to Euclid I, 9



Euclid I, 10

To bisect a given finite straight line.

Construction: Erect a isosceles triangle $\triangle ABC$ upon a given line AB. By Euclid I, 9, the angle $\measuredangle ACB$ is divided in half. The angle bisector intersects the given line AB at the midpoint D (Fig. 10.7a).

Proof: Since the $\triangle ABC$ is isosceles by assumption, $|AC| = |BC|$ applies. The triangles $\triangle AMC$ and $\triangle MBC$ are congruent, since they correspond in the sides AC resp. BC, the common side MC and the included angle $\measuredangle ACD = \measuredangle MCB$. Therefore, $|AM| = |MB|$ must also apply; thus M is the required midpoint of AB.

In the Proclus commentary, the today usual halving method is attributed to Apollonius (Fig. 10.7b). Here, two circles with the centers A, B are intersected with the same radius $|AM| = |MB|$; intersection points are C resp. D. The connecting line CD bisects the given line at the midpoint M. The proof proceeds analogously; the isosceles triangle is here $\triangle ADC$ resp. $\triangle BDC$.

Euclid I, 12

To draw a straight line perpendicular to a given indefinite straight line from a given point without.

Construction (Fig. 10.8): Draw a circle around P, which intersects the line in two points A, B (this is always possible). The center C of the line segment AB is the foot point of the desired perpendicular to the line.

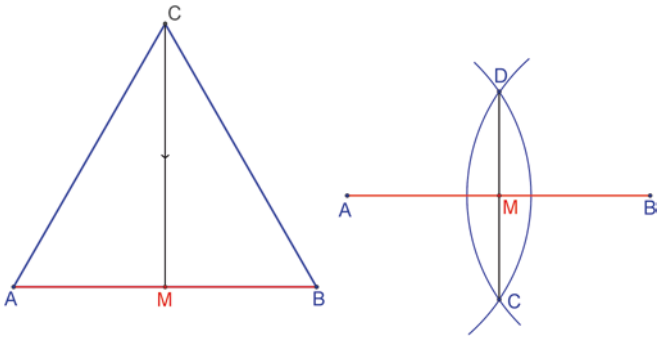
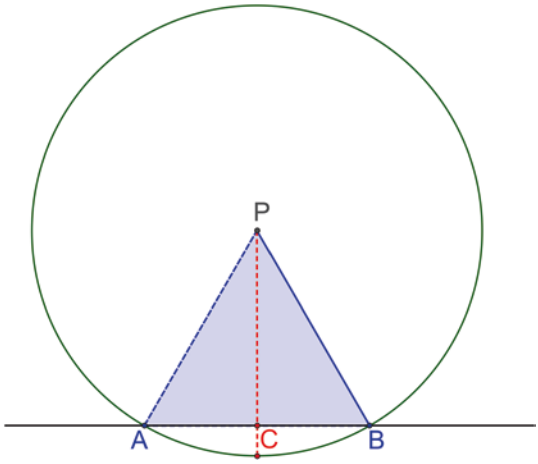


Fig. 10.7 Figure for Euclid I, 10 and Alternative

Fig. 10.8 Figure for Euclid I, 12



Proof: Since the lines $|PA|$ or $|PB|$ represent the circle radius, the triangle $\triangle ABP$ is isosceles. Thus, the base angles $\measuredangle PAB$ and $\measuredangle PBA$ are congruent. The center C divides AB into two congruent line segments $|AC| = |CB|$. Thus, the triangles $\triangle ACP$ or $\triangle CBP$ are also congruent, since they match in two sides and the corresponding interior angle. Since this also makes the angles $\measuredangle ACP$ or $\measuredangle BCP$ congruent and together represent the extended angle $\measuredangle ACB$, each angle is a right angle. This makes PC the sought-after perpendicular from point P to line AB.

Euclid I, 16
If a side of a triangle is produced, the external angle is greater than either of the internal remote angles.

Euclid I, 27
If a straight line meeting two other straight lines makes with them the alternate angles equal, these two straight lines are parallel.

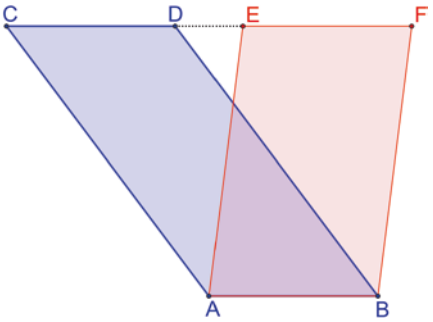


Fig. 10.9 Figure for Euclid I, 35

Euclid I, 28
If a straight line, cutting two other straight lines, makes the external equal to the internal and opposite angle, at the same side of the cutting line, or if it makes the two internal angles at the same side together equal to two right angles, those two straight lines are parallel.

Euclid I, 29
A straight line falling on two parallel straight lines, makes the alternate angles equal to one another; and also the external equal to |PA| |PB| |AC| = |CB| the internal and opposite angle on the same side; and the two internal angles on the same side together equal to two right angles.

Euclid I, 35
Parallelograms on the same base, and between the same parallels, are (in area) equal. (Fig. 10.9).

10.2 From Book II of the Elements

(Parabolic) area layout
Given a rectangle with sides a, b, construct a congruent rectangle that contains a given side c as an extension of a rectangle side (parabolic area layout).

After the construction, it holds that $ab = cx$. The sought distance x satisfies the proportion (Fig. 10.10).

$$\frac{a}{c} = \frac{x}{b}$$

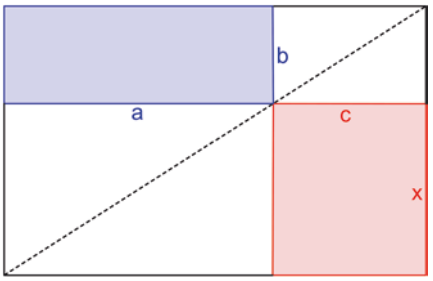


Fig. 10.10 Parabolic area layout

At the same time, a division was carried out geometrically

$$x = \frac{ab}{c}$$

(Elliptical) area layout

The elliptical surface construction is given by Euclid's theorem [Euclid II, 5].

If a line AB is divided by C into equal parts or by D into unequal parts, then the rectangle formed by the unequal parts is equal in area to the square over the two points CB together with the square.

With the designations of Fig. 10.11 the claim is:

$$|AD| \cdot |DB| + |CD|^2 = |BC|^2$$

Draw the square $CILB$ over $|CB|$, draw the lines BI and $DG \parallel CI$ through D, likewise $EH \parallel AB$ through G, likewise $AE \parallel CF$ through A. After construction, rectangle $(CFGD)$ is equal in area to rectangle $(GKLH)$. Addition of rectangle $(DGHB)$ on both sides yields rectangle $(CFHB)$ equal in area to rectangle $(DKLB)$. Since C is the center, $|AC| = |CB|$ and thus rectangle $(AEFC)$ is equal in area to rectangle $(CFHB)$. Addition of rectangle $(CFGD)$ on both sides yields rectangle $(AEGD)$ equal in area to gnomon $CFGKLB$. Rectangle $(AEGD)$ is equal in area to $|AD| \cdot |DG| = |AD| \cdot |DB|$. Thus, gnomon $CFGKLB$ is also equal in area to $|AD| \cdot |DB|$. Addition of square $(FIKG) = |CD|^2$ yields

$$\text{Rectangle}(AEGD) + \text{Square}(FIKG) = |AD| \cdot |DB| + |CD|^2$$

This provides

$$\begin{aligned} |AD| \cdot |DB| + |CD|^2 &= |AC| \cdot |AE| + |CD| \cdot |CF| + |FG|^2 \\ \Rightarrow |AD| \cdot |DB| + |CD|^2 &= |CB| \cdot |CF| + |GH| \cdot |GK| + |FG|^2 \end{aligned}$$

This shows that the gnomon $AEFIKD$ is equal in area to the square $|BC|^2$.

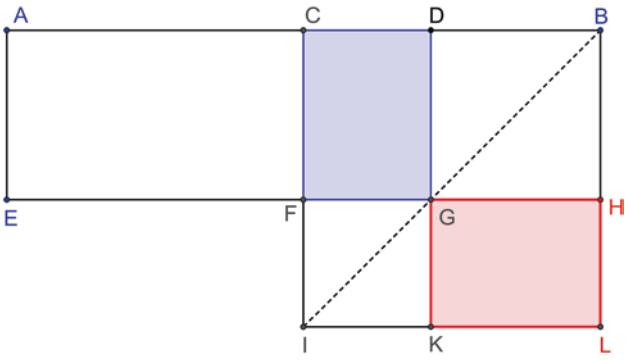


Fig. 10.11 Elliptical area layout

(Hyperbolic) area layout

The hyperbolic surface area is described by the theorem [Euclid II, 6].

If a line segment AB is divided into equal parts by C and produced by the line segment BD, then the rectangle formed by the produced line segment and the added line segment, together with the square upon half the line segment CB, is equal in area to the square upon the sum of half the line segment and the additional line segment BD. With the notation of Fig. 10.12 the claim is

$$|AD| \cdot |DB| + |BC|^2 = |CD|^2$$

After construction, it follows that

$$\begin{aligned}
&|AD| \cdot |DB| + |BC|^2 = 2|CB| \cdot |BG| + |FG|^2 \\
\Rightarrow &|AD| \cdot |DB| + |BC|^2 = (|AC| + |CB| + |BD|) \cdot |DB| + |BC|^2 = (2|CB| + |BD|) \cdot |BG| + |BC|^2 \\
\Rightarrow &|AD| \cdot |DB| + |BC|^2 = |CB| \cdot |BG| + \underbrace{|CB| \cdot |BG|}_{|GH| \cdot |KL|} + |BD| \cdot |BG| + |BC|^2 = |CD|^2
\end{aligned}$$

Here the gnomon *with* the square of the corners $|BD|^2$ is equal in area to the square $|CD|^2$.

The theorems of so-called *geometric algebra* discussed here are currently the subject of a fierce debate among historians of mathematics. The followers of S. Unguru consider the theorems in Book II to be pure geometry theorems and deny the Greeks the ability to engage in algebra.

Euclid II, 12

In Book II, 12 or II, 13, Euclid proves a theorem analogous to the law of cosine (Fig. 10.14)

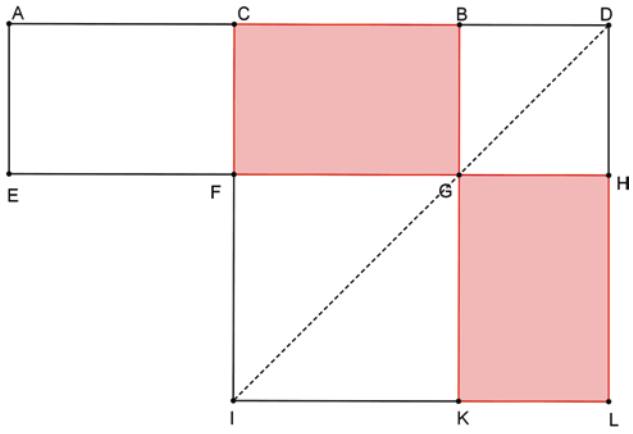


Fig. 10.12 Hyperbolic area layout

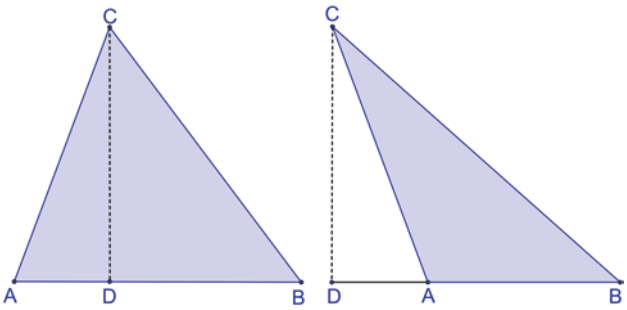


Fig. 10.13 Figure for Euclid II, 12

Case A) of the obtuse triangle:
If $\measuredangle BAC$ is the obtuse angle and D is the foot point of C on the line AB. Then according to Euclid II, 4

$$|DB|^2 = (|DA| + |AB|)^2 = |DA|^2 + |AB|^2 + 2|DA||AB|$$

$$|CD|^2 + |DA|^2 = |AC|^2 \ \therefore \ |CD|^2 + |DB|^2 = |BC|^2 \Rightarrow |AC|^2 - |DA|^2 = |BC|^2 - |DB|^2$$
$$\Rightarrow |DB|^2 = |BC|^2 - |AC|^2 + |DA|^2$$

According to Pythagoras, in the right-angled triangles $\triangle DAC$ and $\triangle DBC$
Setting equal gives

$$|BC|^2 - |AC|^2 + |DA|^2 = |DA|^2 + |AB|^2 + 2|DA||AB|$$
$$|BC|^2 = |AB|^2 + |AC|^2 + 2|DA||AB|$$

Case B) of the acute-angled triangle

If D is the foot point of the altitude, then in the right-angled triangles $\triangle ADC$ and $\triangle DBC$

$$|CD|^2 + |DA|^2 = |AC|^2 \ \therefore \ |CD|^2 + |DB|^2 = |BC|^2 \Rightarrow |AC|^2 - |DA|^2 = |BC|^2 - |DB|^2$$
$$\Rightarrow |DB|^2 = |BC|^2 - |AC|^2 + |DA|^2$$

Furthermore, it holds

$$|DB|^2 = (|AB| - |AD|)^2 = |AB|^2 + |AD|^2 - 2|AB||AD|$$

Setting equal again gives

$$|BC|^2 - |AC|^2 + |AD|^2 = |AB|^2 + |AD|^2 - 2|AB||AD|$$
$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AD|$$

Both cases can be summarized as

$$|BC|^2 = |AB|^2 + |AC|^2 \pm 2|AB||AD|$$

In the case of a right-angled triangle, the projection of one leg onto the other is zero (here $|AD| = 0$) and the Pythagorean theorem results as a special case.

Euclid II, 14

The task is to find the square with the same area as a given area.

Construction (Fig. 10.14): Let $BCDE$ be a rectangle congruent to the given surface area. If $|BC| = |BE|$, then the rectangle is a square and the task is solved. Let BE be the longer side of the rectangle. $|BE|$ is extended by $|ED|$ to the point F. The midpoint of the line BF is G. A semicircle is erected with the center G upon BF. The perpendicular in the point E to BF intersects the semicircle in the point H. $|EH|$ is the desired side length of the congruent square.

Proof: Since the point G divides the BF in half and the point divides the distance unequally, according to [Euclid II, 5]

$$|BE||EF| + |GE|^2 = |GH|^2$$

In the right-angled $\triangle\, GEH$ it applies $|GH|^2 = |GE|^2 + |EH|^2$. Substituting into the previous equation gives

$$|BE||EF| + |GE|^2 = |GE|^2 + |EH|^2$$

Removing the square $|GE|^2$ gives $|BE||EF| = |EH|^2$. Because $|EF| = |ED| = |BC|$ this is the claim

$$|BE||BC| = |EH|^2$$

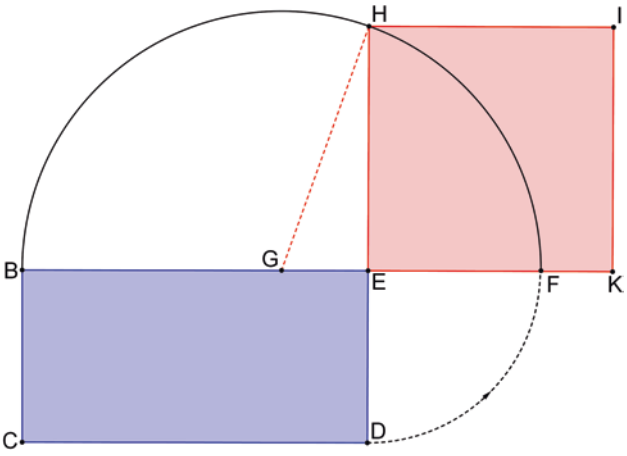


Fig. 10.14 Figure for Euclid II, 12

10.3 The Circle Theorems in Book III

The Chord Theorem [Euclid III, 35] (Fig. 10.15).

If two chords AB and CD of a circle intersect in a point P, then the product of the chord segments is constant; i.e. it applies

$$|AP| \cdot |BP| = |CP| \cdot |DP|$$

The similarity proof proceeds as follows: The triangles ACP and PBD are similar because they match at the opposite angle at P. Furthermore, the angles CAB and CBD are congruent because these are at the common chord CB. Because of the similarity, the claim follows:

$$\frac{|AP|}{|CP|} = \frac{|DP|}{|BP|} \Rightarrow |AP| \cdot |BP| = |CP| \cdot |DP|$$

The secant theorem (Fig. 10.16).

If two secants intersect at a point P outside a circle and A,B and C,D are the points of intersection with the circle, then the following is true:

$$|AP| \cdot |BP| = |CP| \cdot |DP|$$

The proof of similarity is relatively simple here as well: the $\triangle APD$ and CBP are similar because they match in angle at P. Furthermore, the $\measuredangle PAD$ and BCP are congruent because they are angles at the common chord BD. Because of the similarity, the following claim follows:

$$\frac{|AP|}{|CP|} = \frac{|DP|}{|BP|} \Rightarrow |AP| \cdot |BP| = |CP| \cdot |DP|$$

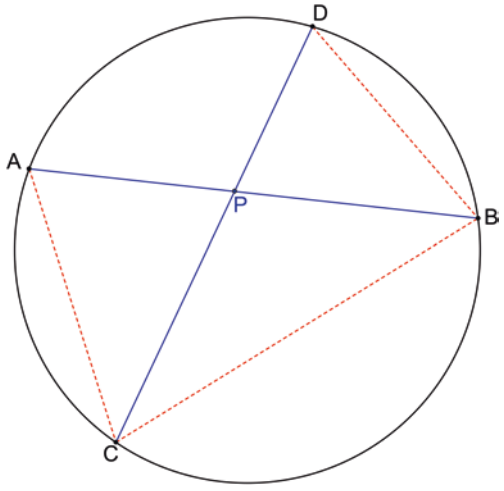


Fig. 10.15 To the Chord Theorem

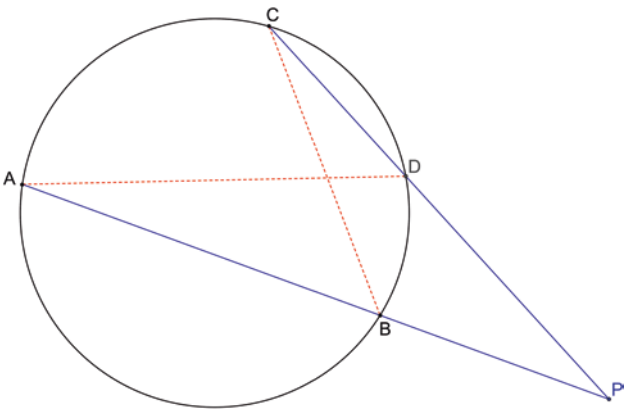


Fig. 10.16 To the secant theorem

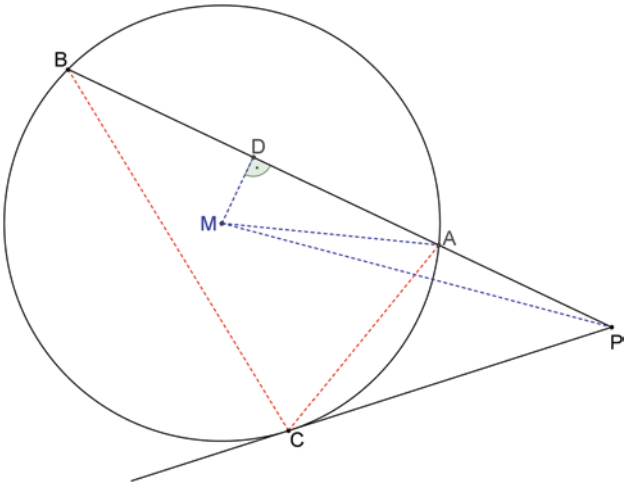


Fig. 10.17 To the chord-tangent theorem

The tangent-secant theorem [Euclid III, 36] (Fig. 10.17). If from a point without a circle two straight lines be drawn to it, one of which is a tangent to the circle, and the other cuts it; the rectangle under the whole cutting line and the external segment is equal to the square of the tangent

If a tangent and secant intersect at a point P outside of a circle and A, B are the points of intersection and C is the point of tangency, then the following is true:

$$|AP| \cdot |BP| = |CP|^2$$

Similarity proof: The triangles $\triangle APC$ and CBP are similar, since they match in angle at P. Furthermore, the $\measuredangle PAC$ or $\measuredangle BCP$ are congruent, since the

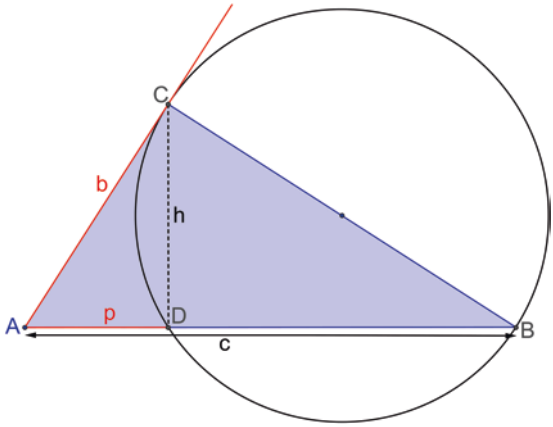


Fig. 10.18 Special case of the secant-tangent theorem

tangent-chord angle $\measuredangle BCP$ is congruent to the circumference angle over the tendon BC. Because of the similarity, the claim follows:

$$\frac{|AP|}{|CP|} = \frac{|PC|}{|PB|} \Rightarrow |AP| \cdot |BP| = |CP|^2$$

Euclid provides a congruence proof here.

Pythagorean Theorem as a special case of the tangent-secant theorem (Fig. 10.18):

It applies according to the secant-tangent theorem

$$|AC|^2 = |AD| \cdot |AB| \Rightarrow b^2 = cp$$

This is the Pythagorean Theorem in $\triangle ABC$, which is right-angled, since the tangent section AC is perpendicular to the diameter BC.

Altitude theorem as a special case of the Pythagorean theorem:

The chord with the sections p, q contains the center of the circle. Then, according to the Pythagorean theorem

$$h^2 = pq \Rightarrow h = \sqrt{pq}$$

This is the altitude theorem in the inscribed triangle, which is right-angled according to Thales.

10.4 Perfect and Friendly Numbers

The concept of number is only introduced by Euclid in Book VII. Definition (2) in the Heiberg-Menge edition[11] simply states: *Numerus autem est multitudo ex unitatibus composita* (A number consists of a multitude of units). Boethius already

[11] Euclidis Opera Omnia, Ed. Heiberg, Menge, Band II, Teubner 1884.

takes into account sizes with units: *Numerus est unitatis collectio vel quantitis acervus ex unitatibus profusus* (A number is a collection of units or a quantity of sizes, profuse in units). Leonardo of Pisa includes the possibility of reaching infinity by successively adding the unit to his definition. At the beginning of his *Liber Abbaci* he writes:

> Nam numerus est unitatum perfusa collectio sive congregatio unititatum, que per suos in infinitum ascendit gradus.

This definition of the number is kept in essence throughout the Middle Ages, even though it does not include fractions.

Definition (VII, 22) states: *A perfect number is one that is equal to its parts*. Greek mathematics only knew of real factors t of a number n; these are all factors with $1 \leq t < n$. In the modern sense, every number is a factor of itself. The sum of the factors of a perfect number n is therefore equal to $2n$ or n for proper factors.

The theorem [Euclid IX, 36] states: If one adds up, starting with one, all the summands, doubling each time, until a prime number results, then the product of the sum and its last summand is a perfect number. So the claim is that the following number is perfect

$$\left(\underbrace{1+2+2^2+\cdots+2^{n-1}}_{prim}\right)2^{n-1}$$

Using the geometric series, one obtains the formula $2^{n-1}(2^n-1)$. For $n \in \{2,3,5,7\}$ one obtains the perfect numbers already known to the Greeks, 6, 28, 496 and 8128. As an example, let $496 = 2^4 \cdot 31$ be chosen. The number of factors of 496 is equal to 10; the corresponding sum of factors yields

$$1+2+4+8+16+31+62+124+248+496 = 2 \cdot 496$$

The next two perfect numbers are 33,550,336 and 8,589,869,056.

L. Euler could show that every even perfect number necessarily has the form of Euklid. All perfect numbers found so far are even. Whether there are also odd perfect numbers is still unresolved.

The derivation given above is probably from the Pythagorean *Archytas.* This shows that Euklid drew on the knowledge of the Pythagoreans here. This knowledge about perfect numbers has given rise to numerous speculation in antiquity and in the Middle Ages. So Augustine (354–430) proclaimed in his work *The City of God:*

> God could have created the world in one moment. However, he chose the 6 days to show the perfection of the universe.

Augustine made the comment that the number 6 is not perfect because God chose it, but rather its perfection is inherent to it:

> The 6 is perfect in and of itself, but not because God created all things in 6 days. On the contrary: God created all things in 6 days because this number is perfect. And it would remain perfect even if the work of the 6 days did not exist.

Number pairs with the property that one number represents the true sum of divisors of the other and vice versa are called befreundet. The Pythagoreans knew the amicable pair (220; 284); the friendship of the numbers is recognized by means of the sum of all true divisors; here 284 or 220.

For the imperfect numbers, Nikomachos and Theon of Smyrna added the following definition: Numbers whose true sum of divisors is less than the number are called *defizient;* correspondingly with a greater sum of divisors, *abundant.* Nikomachos writes.

> While these latter two types of numbers resemble human vices in that they, like vices, are very common and do not submit to any particular order, the perfect numbers behave like virtue in that they preserve the right measure, the "mean" between abundance and deficiency.

Alcuin, a teacher at the court of Charlemagne, also believes that the number 6 determines the universe because it is perfect. The second creation of humanity came about from the deficient number 8; this is namely the number of souls on board the Noah's Ark. Since the number 8 is imperfect, this second creation is less successful than the first, resulting from the perfect number 6.

Amicable Numbers

With the help of the Euclidean formula for perfect numbers, Thābit ibn Qurra was even able to formulate an explicit formula for friendly numbers. If the following three numbers are prime

$$p = 3 \cdot 2^n - 1;\ q = 3 \cdot 2^{n-1} - 1;\ r = 9 \cdot 2^{2n-1} - 1$$

then one obtains a pair of friendly numbers from

$$a = 2^n pq;\ b = 2^n r$$

In the case of $n = 2$ one obtains the prime numbers $p = 11;\ q = 5;\ r = 71$; this provides the (already known) amicable numbers $a = 220;\ b = 284$. For $n = 3$ the algorithm fails, since for $r = 287 = 7 \cdot 41$ no prime number results. For $n = 4$ and $n = 7$ the pairs (17,296; 18,416) and (9,363,584; 9,437,056) result. More pairs of amicable numbers are

(1184; 1210), (2620; 2924), (5020; 5564), (6232; 6368)

10.5 The Euclidean Algorithm

Euclid's algorithm (Euclid VII, 2) determines the greatest common divisor (*gcd*) of two natural numbers a, b by alternately subtracting the smaller number from the larger one. If the subtraction of the smaller number from the larger one ends with two equal lines, then the common length of these lines is the *gcd(a,b).* At the same time, it is proved that the ratio $\frac{a}{b}$ is rational. The repeated subtraction is usually replaced by division.

Example: The calculation of the *gcd (81,51)* is carried out according to the following algorithm

$$
\begin{aligned}
81 &= 51 \cdot 1 + 30 \\
51 &= 30 \cdot 1 + 21 \\
30 &= 21 \cdot 1 + 9 \\
21 &= 9 \cdot 2 + 3 \\
9 &= 3 \cdot \boxed{3}
\end{aligned}
$$

In the last step, the line of length 3 *without remainder* can be subtracted from the line 9 three times. Thus *gcd (81, 51)=3.* The procedure is illustrated geometrically in Fig. 10.19.

Continued Fraction

Euclid's method can also be used to calculate continued fractions, which cannot be found in Greek literature.

The rationality of the number $\frac{81}{51}$ is shown by a *finite* continued fraction.

$$
\begin{aligned}
81 = 51 \cdot 1 + 30 &\Rightarrow \tfrac{81}{51} = 1 + \tfrac{30}{51} \\
51 = 30 \cdot 1 + 21 &\Rightarrow \tfrac{51}{30} = 1 + \tfrac{21}{30} \\
30 = 21 \cdot 1 + 9 &\Rightarrow \tfrac{30}{21} = 1 + \tfrac{9}{21} \\
21 = 9 \cdot 2 + 3 &\Rightarrow \tfrac{21}{9} = 2 + \tfrac{3}{9} \\
9 = 3 \cdot \boxed{3}
\end{aligned}
$$

Composing the fractions results in the finite continued fraction

$$
\begin{aligned}
&\frac{81}{51} = 1 + \frac{30}{51} = 1 + \frac{1}{\frac{51}{30}} = 1 + \frac{1}{1 + \frac{21}{30}} = 1 + \frac{1}{1 + \frac{1}{\frac{30}{21}}} \\
\Rightarrow &\frac{81}{51} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{9}{21}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3}}}} = [1; 1, 1, 2, 3]
\end{aligned}
$$

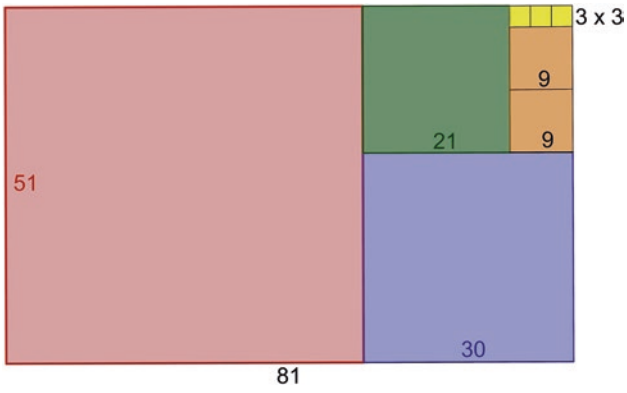


Fig. 10.19 Euclidean Algorithm (geometrically illustrated)

Continued Fraction of an Irrational Number
The continued fraction method can also be extended to irrational numbers. In modern notation, one obtains successively:

$$\begin{aligned}
\sqrt{7} = 2 + \tfrac{1}{x_1} &\Rightarrow x_1 = \tfrac{1}{\sqrt{7}-2} = \tfrac{\sqrt{7}+2}{3} &&\Rightarrow [x_1] = 1\\
x_1 = 1 + \tfrac{1}{x_2} &\Rightarrow x_2 = \tfrac{1}{x_1-1} = \tfrac{\sqrt{7}+1}{2} &&\Rightarrow [x_2] = 1\\
x_2 = 1 + \tfrac{1}{x_3} &\Rightarrow x_3 = \tfrac{1}{x_2-1} = \tfrac{\sqrt{7}+1}{3} &&\Rightarrow [x_3] = 1\\
x_3 = 1 + \tfrac{1}{x_4} &\Rightarrow x_4 = \tfrac{1}{x_3-1} = \sqrt{7}+2 &&\Rightarrow [x_4] = 4\\
x_4 = 1 + \tfrac{1}{x_5} &\Rightarrow x_5 = \tfrac{1}{x_4-1} = \tfrac{\sqrt{7}+2}{3} = x_1
\end{aligned}$$

The brackets [...] mean a rounding down to the next smaller integer. Since the fifth denominator is equal to the first, a period of length 4 is obtained here. Thus, the periodic continued fraction $\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, \ldots] = \left[2; \overline{1, 1, 1, 4}\right]$.

Extended Euclidean algorithm
According to a theorem of J.G. Bachet (de Meziriac), the greatest common divisor *gcd (x, y)* of two integers x, y can be written as a linear combination of these numbers

$$ax + by = gcd(x, y);\ a, b \in \mathbb{N}$$

The corresponding multiples *a, b* can be determined with the extended Euclidean algorithm. We determine $gcd(81, 51)$.

$$\begin{aligned}
81 &= 51 \cdot 1 + 30 &&\Rightarrow 30 = 81 - 51 \cdot 1\\
51 &= 30 \cdot 1 + 21 &&\Rightarrow 21 = 51 - 30 \cdot 1\\
30 &= 21 \cdot 1 + 9 &&\Rightarrow 9 = 30 - 21 \cdot 1\\
21 &= 9 \cdot 2 + 3 &&\Rightarrow 3 = 21 - 9 \cdot 2\\
9 &= 3 \cdot \boxed{3}
\end{aligned}$$

By working backwards, one obtains successively

$$\begin{aligned}
3 &= 21 - 9 \cdot 2 = 21 - (30 - 21 \cdot 1) \cdot 2 = 21 \cdot 3 - 30 \cdot 2\\
\Rightarrow 3 &= (51 - 30 \cdot 1) \cdot 3 - 30 \cdot 2 = 51 \cdot 3 - 30 \cdot 5\\
\Rightarrow 3 &= 51 \cdot 3 - (81 - 51 \cdot 1) \cdot 5 = 51 \cdot 8 - 81 \cdot 5\\
3 &= gcd(81, 51) = 51 \cdot 8 + 81 \cdot (-5)
\end{aligned}$$

This is *one* of the possible linear combinations; another is (see also 21.6)

$$3 = gcd(81, 51) = 51 \cdot 35 + 81 \cdot (-22)$$

10.6 Euclid's Theorem on Primes

Euclid's famous theorem (IX, 20) on the infinity of the set of prime numbers, also called the prime number theorem, is in the form

There are more prime numbers than any number of given prime numbers.

Euclid's Proof
Under the assumption that there are only finitely many different prime numbers $p_i (1 \leq i \leq n)$ one forms the term

$$N = p_1 \cdot p_2 \cdot p_3 \cdots p_n + 1 = 1 + \prod_{i=1}^{n} p_i \quad (*)$$

For the number thus formed N there are two possibilities.

Case 1) N is itself a prime number. Because of $N > p_i$ N is a new further prime number $p_j (j > n)$. This gives a contradiction to the assumption.

In case 2) N is composite and has at least one prime factor $q|N$. According to the assumption $q = p_i$. must hold. Therefore $q|p_1 \cdot p_2 \cdot p_3 \cdots p_n$. also holds. According to [Euclid VII, 5] a common factor of two numbers must also divide their difference $t|a \wedge t|b \Rightarrow t|(a - b)$. Therefore it follows

$$\left.\begin{array}{c} q|N \\ q|p_1 \cdot p_2 \cdot p_3 \cdots p_n \end{array}\right\} \Rightarrow q|1 \Rightarrow q = 1$$

In both cases one obtains a contradiction. The set of prime numbers is therefore infinite. Euclid's proof is generalizable to the residue classes (± 1 *mod* 4). According to W. Dunham Euclid's proof has been chosen as one of the top 10 most beautiful proofs!

10.7 The Parallel Axiom

Euclid formulates the parallel postulate (P) as the fifth postulate in Book I of the Elements:

> If a line intersects two other lines so that the sum of the angles on the same side is less than two right angles, then the two lines intersect on the side where the two angles with the sum less than two right angles lie.

Often the axiom is quoted in the formulation of J. Playfair (eighteenth century):

For every point P outside a line $g (P \notin g)$ there is **exactly** one parallel $h \parallel g$ to the given line with $(P \in h)$.

A consequence of the parallel postulate is the interior angle theorem in a triangle [Euclid I, 32].

Interior Angle Sum
Proof: In $\triangle ABC$, the side AB is produced to point D; at point B, the parallel $BE \parallel AC$ is drawn (Fig. 10.20).

Since the parallel pair AC and BE is cut by the line BC, the alternate angles $\measuredangle ACB = \measuredangle CBE$ are congruent. The line AD also cuts the parallel pair; thus, the

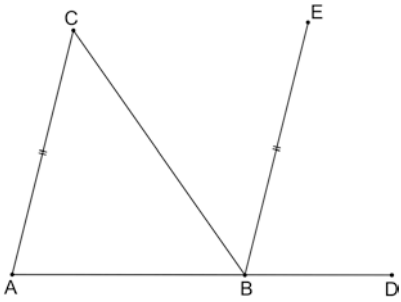


Fig. 10.20 Figure for Euclid I, 32

angle sum $\measuredangle CAB + \measuredangle ABE = 2R$ and the angles $\measuredangle CAB = \measuredangle EBD$ are congruent. Since angle $\measuredangle ABD$ is a straight angle, it follows that

$$\angle ABD = \angle ABC + \angle CBE + \angle AED = \angle ABC + \angle ACB + \angle CAB = 2R$$

This shows that the interior angle sum in a triangle is equal to two rights. Furthermore, it follows that the exterior angle $\angle CBD$ is equal to the angle sum of the two non-adjacent interior angles

$$\angle CBD = \angle CAB + \angle ACB$$

Euclid here takes from the observation that the line BE does not run inside the triangle ABC.

Properties of parallels

1. If two lines $g \neq h$ have a common perpendicular ℓ, they are parallel.
   Converse to (1): If two lines $g \neq h$ are parallel, they have a common perpendicular ℓ.
2. If two parallel lines $g \parallel h$ are cut by a third line, then the alternate angles are congruent (Euclid I, 28)
   Converse to (2): If the alternate angles are congruent when two lines g, h are cut by a third line, then the lines are parallel $g \parallel h$ (Euclid I, 27).
3. If two lines are parallel and two perpendiculars are dropped from one line to the other line, then the lengths of the perpendiculars are congruent.
   Converse to (3): If two points A,B of a line g have the same distance from the line h and they lie on the same side of h, then the lines are parallel.

10.8 Equivalent Postulates to the Parallel Axiom

One thing that stands out in the first book of Elements is that the first 29 theorems are formulated without the parallel axiom; all the later theorems, however, use the axiom. Obviously, Euclid wanted to bring this axiom as late as possible. The first scholar that we know of being dissatisfied with the parallel axiom was *Poseidonios*

(135–51 BC). He defined the parallel as a line with always the same distance and tried to prove it within the framework of Euclidean theorems.

In the following centuries, mathematicians have repeatedly tried to formulate equivalent geometric theorems to the parallel axiom and thus make them provable. Such analogous postulates are:

- If a line intersects one line of a pair of parallels, then the other parallel is also cut (Proclus around 440)
- A quadrilateral for which two opposite sides each form a perpendicular to a base is a rectangle (G. Vitale 1680, *G. Saccheri* 1733)
- For a given triangle, a similar triangle can always be constructed (*J. Wallis* 1663)
- Every quadrilateral with 3 right angles is a rectangle (*J.H. Lambert* 1766)
- Every line that goes through a point inside an angle field intersects at least one leg of the angle (*J.F. Lorentz* 1791)
- A circle can always be drawn through three points that do not lie on a line (*J.M. Legendre*)
- There are similar (not congruent) triangles of any size (*J. Wallis* 1693)

A) The Saccheri Quadrilateral

In his work *Euclides ab Omni Naevo Vindicatus* (1733), the Jesuit G. Saccheri attempted to prove the parallel axiom using the so-called Saccheri quadrilateral. This Saccheri quadrilateral (which was already considered by Omar Khayyām in the eleventh century) consists of the base line AB and the two congruent opposite sides AD and BC, both of which represent a perpendicular on AB. For symmetry reasons, the two remaining angles α, β are congruent. Saccheri was able to reduce the case $\alpha, \beta > R$ to a contradiction. The contradiction in the case $\alpha, \beta < R$ could not be proven(Fig. 10.21).

The congruence $\alpha = \beta$ can be proven as follows. First, $\triangle ABD$ and $\triangle ABC$ are congruent, since they are at right angles and the sides $|AD| = |BC|$ and AB match. Thus, the diagonals $|AC| = |BD|$ are also congruent. It follows that $\triangle ACD$ and $\triangle DBC$ are also congruent. They match in all three sides: CD is a common side and $|AC| = |BD|$ and $|AD| = |BC|$ according to the assumption.

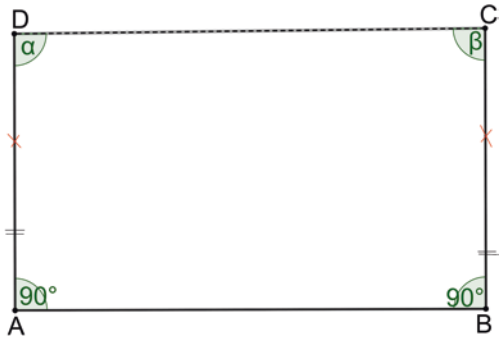


Fig. 10.21 Saccheri quadrangle

It can be shown that the sum $\alpha + \beta$ of the remaining angles is equal to the angle sum in the triangle. Given is the triangle ABC with the midpoints D resp. E. The connecting line DE is drawn through these midpoints. From the vertices A,B the lines are drawn to the connecting line; the points of intersection are F resp. G. The point of intersection of the line from vertex C to FG is to be the point H; e.g. according to the given conditions, H lies between D and E (Fig. 10.22).

The triangles $\triangle ADF$ and $\triangle DHC$ are congruent, since $|AD| = |DC|$ and the triangles are right angled and the opposite angle at D match. Analogously, the triangles $\triangle CHE$ and $\triangle EGB$ are also congruent, since $|CE| = |EB|$ and the triangles are right-angled and the apex angle at E match. Consequently, it follows that

$$\left.\begin{array}{l} |AF| = |CH| \\ |CH| = |GB| \end{array}\right\} \Rightarrow |AF| = |GB|$$

The quadrilateral $ABFG$ is therefore a Saccheri quadrilateral with the base line FG and the two right angles at F resp. G.

Because of the aforementioned congruences, the angle sum at A and B can be written as

$$\measuredangle FAB + \measuredangle ABG = \underbrace{\measuredangle FAD}_{\measuredangle DCH} + \measuredangle DAB + \measuredangle ABE + \underbrace{\measuredangle EBG}_{\measuredangle HCE} = \measuredangle DAB + \measuredangle ABE + \measuredangle DCE$$

Thus, the angle sum at A and B is equal to the angle sum of the triangle ABC. The requirement that a Saccheri quadrilateral is a rectangle is therefore equivalent to the validity that the sum of the triangle angles is equal to two right angles.

B) The Wallis Triangle

J. Wallis proposed the following construction (Fig. 10.23). Given is a right-angled triangle ABC. On the line BC an arbitrary point Q is chosen and the perpendicular

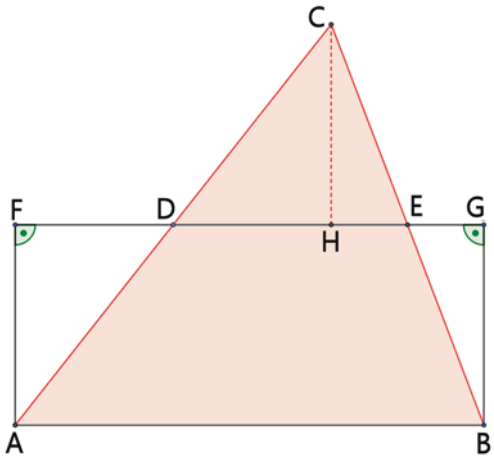


Fig. 10.22 Saccheri quadrangle (2)

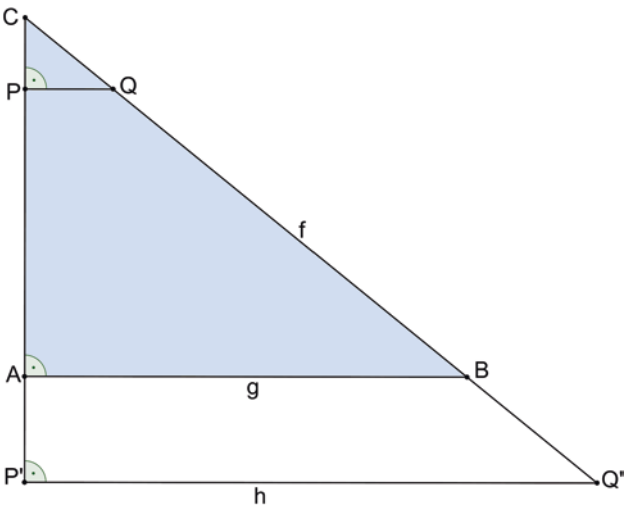


Fig. 10.23 To the Wallis triangle

is dropped upon the line AC; the foot point is P. According to the Archimedean Axiom, there exists to two lines $|CP| < |CA|$ a multiple of the smaller one so that it holds

$$|CP| < |CA| \Rightarrow n|CP| > |CA|$$

If you apply the multiple distance $n|CP|$ from the point C, you get the point P'. The points C, P' can now be used to supplement a similar triangle $\triangle P'Q'C$ from $\triangle ABC$. Because of the similarity, the angles $\measuredangle ABC$ and $\measuredangle P'Q'C$ are congruent. Thus, the line $f = BC$ intersects the lines $g = AB$ and $h = P'Q'$ at congruent alternate angles. The ability to construct a similar triangle to any triangle is therefore equivalent to the validity of the parallel axiom.

C) Legendre's Postulate

If one assumes that three arbitrary points determine a circle, then the existence of a parallel can be proven.

This can be made plausible with the following construction (after Steiner) (Fig. 10.24):

Given is a line AB and a point $P \notin AB$ outside with $|AP| \neq |BP|$. Then, according to the assumption, a circle can be constructed through the three points A, B and P. Now a circle is drawn with center B with the radius $r = |AP|$. The latter intersects the first circle in another point $Q \neq P$, since, according to the assumption, $|PA| \neq |PB|$ applies. The points P, Q uniquely determine the parallel $PQ \parallel AB$.

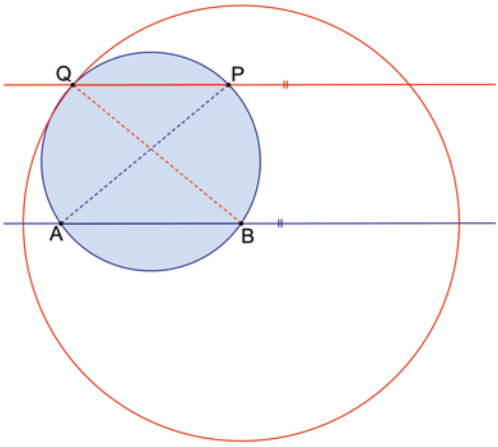


Fig. 10.24 To Legendre's Postulate

D) Legendre's Theorem

A.-M. Legendre proves the following theorem: The sum of the angles of a triangle is less than or equal to two rights ($2R$).

A. Legendre proves this in two steps (Fig. 10.25):

1. Step: Every triangle can be mapped to another one with the same sum of angles, where one angle is halved.

Given the arbitrary triangle $\triangle ABC$ with the angles α, β, γ with $\beta \leq \gamma$. The midpoint of the side BC is D. The line AD is produced to the point E, so that $|AD| = |DE|$ applies. The triangle $\triangle ABE$ thus formed has the angles α', β', γ'. The $\triangle ADC$ and $\triangle DBE$ are congruent, since they coincide in the opposite angle at D and in the congruent sides $|CD| = |DB|$ or $|AD| = |DE|$. From the congruence of the angles follows

$$\gamma = \beta' - \beta \Rightarrow \beta' = \gamma + \beta$$
$$\alpha - \alpha' = \gamma' \Rightarrow \alpha' = \alpha - \gamma'$$

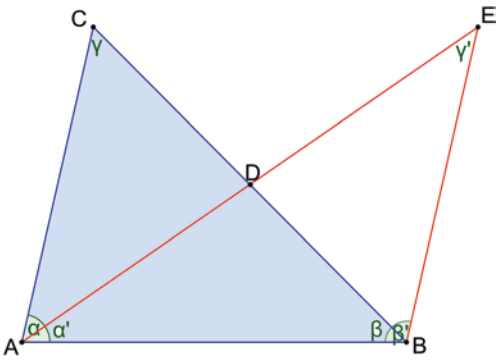


Fig. 10.25 To the Legendre theorem

The angle sum of $\triangle ABE$ agrees with $\triangle ABC$ because of

$$\alpha' + \beta' + \gamma' = \alpha - \gamma' + \gamma + \beta + \gamma' = \alpha + \beta + \gamma$$

Since the larger side also faces the larger angle, it follows that

$$\beta \leq \gamma \Leftrightarrow \underbrace{|AC|}_{|BE|} \leq |AB| \Leftrightarrow \underbrace{|BE|}_{2|BD|} \leq |AB| \Leftrightarrow \alpha' \leq \gamma'$$

This results in the claim

$$2|BD| \leq |BC| \Leftrightarrow |BD| \leq \frac{1}{2}|BC| \Leftrightarrow \alpha' \leq \frac{\alpha}{2}$$

2. Step: The statement [Euclid I, 17] says that the sum of two inner angles in a triangle is smaller than two right angles ($2R$). It is assumed that the angle sum of a triangle is greater than two right angles. With the arbitrarily small angle surplus ε it then applies

$$\alpha + \beta + \gamma = 2R + \varepsilon > 2R$$

According to the Archimedes axiom, the angle α can be reduced by r-fold application of the first step to $\frac{\alpha}{2^r} < \varepsilon$. Because of the constancy of the angle sum, it then applies

$$\alpha^* + \beta^* + \gamma^* = 2R + \varepsilon \ \therefore \ \alpha^* = \frac{\alpha}{2^r} < \varepsilon$$

Insertion shows that in this triangle it applies

$$2R + \varepsilon = \frac{\alpha}{2^r} + \beta^* + \gamma^* \leq \varepsilon + \beta^* + \gamma^* \Rightarrow \beta^* + \gamma^* \geq 2R$$

This is a contradiction to [Euclid I, 16]; therefore the angle sum is smaller or equal to $2R$.

E) Postulate of W. Bolyai

W. Bolyai (1832) takes up the postulate of Legendre and proves it in his own way:

A circle is uniquely determined by three (different) points that do not lie on a line.

Bolyai leads to the following proof (Fig. 10.26): Given are two lines AA' and BB', of which exactly one is perpendicular to AB; the angle α, lies between AA' and AB. Be M an arbitrary point of the line AB. M is mirrored at the line AA' (to point M') and line BB' (to point M''). If α is acute, the three points M, M', M'' are not collinear and determine according to assumption a circle. The existence of this circle requires that AA' and BB' intersect . If AA' and BB' did not intersect, they would be parallel and the point of intersection (= center of the circle) does not exist.

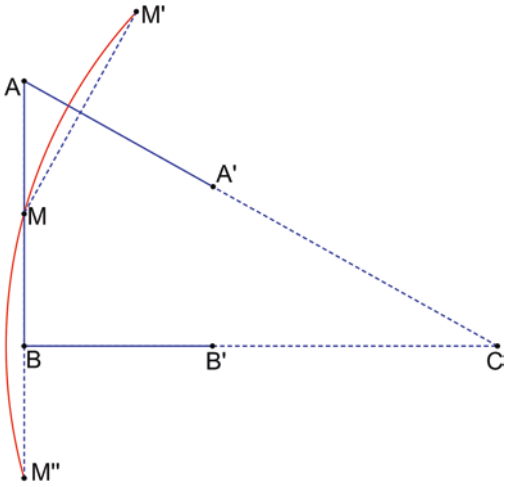


Fig. 10.26 To Bolyai's postulate

The claim made above that C is the center of the circle (M, M', M'') can be shown as follows: Since the point pairs (M, M') and (M, M'') have the same distance from the symmetry lines AA' resp. BB', this also applies to the intersection $C = AA' \cap BB'$; thus

$$|MC| = \left|M'C\right| = \left|M''C\right|$$

A multitude of attempts to prove the parallel postulate cannot be discussed here. A comprehensive presentation can be found in Rosenfeld[12].

10.9 Book of Area Divisions

Euclid's *Book of Divisions* is been lost, but can be partially reconstructed from Arabic sources; it contains a number of more challenging problems on the division of a figure into areas.

Theorem 19 (simplified)
Given is an acute-angled triangle ABC and a point D on the triangle side AC. To find the transversal DF of the triangle, which halves the area of ABC.

Construction (Fig. 10.27): The midpoint E of the side AB is constructed, E is connected to D, and the parallel through the vertex point C ($CF \,||\, ED$) is drawn. The transversal DF halves the area of $\mathcal{F}(ABC)$.

[12] Rosenfeld B.A.: A History of Non-Euclidean Geometry, Springer 1988, pp. 35–109.

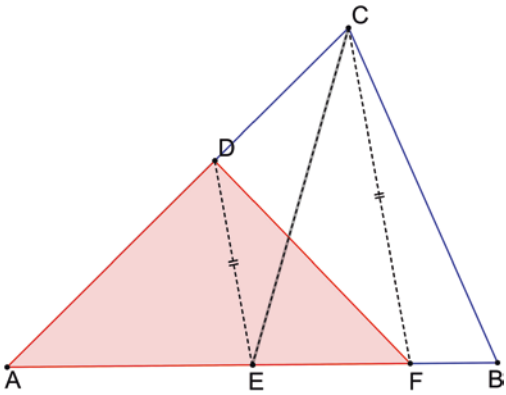


Fig. 10.27 Figure for Theorem 19

Proof: Because of $|AE| = |EB|$the triangle AEC is equal in area to the triangle EBC. Because of the halved baseline, the triangle $\triangle AEC$ has half the area of triangle $\triangle ABC$. Since $\triangle AEC$ can be divided into the partial triangles $\triangle AED$ and $\triangle ECD$, it follows that

$$\mathcal{F}(AED) + \mathcal{F}(ECD) = \frac{1}{2}\mathcal{F}(ABC)$$

According to the construction, triangle EFD is also equal in area to ECD. Thus it applies that

$$\mathcal{F}(AED) + \mathcal{F}(EFD) = \frac{1}{2}\mathcal{F}(ABC)$$

Since the triangles $\triangle AED$ and $\triangle EFD$ together result in triangle $\triangle AFD$, $\triangle AFD$ has half the area of $\triangle ABC$. The task is solvable if the parallel to DE through C intersects the triangle side AB. This is the case if $CD<AD$ applies.

Theorem 28

Given is the surface $ADBC$, consisting of the (acute-angled) $\triangle ABC$ and the circle segment ABD, whereby D is the center of the circle arc over AB.

Required is a line through D, which halves the surface area.

Construction (Fig. 10.28): Let E be the midpoint of the side AB. The parallel to the line CD through E intersects the triangle at the point F and the circle arc at the point G. The connecting line FD is the sought line.

Proof: The line segment $CE \cup ED$ halves the given area, since E and D are the midpoints of AB and the arc, respectively. The $\triangle DCE$ is congruent to $\triangle DCF$, since both have the same base DC and height because of $GF \parallel DC$. The area $DBCE$ contains exactly half the content by assumption; thus the equality of areas follow

$$\mathcal{F}(DBCE) = \mathcal{F}(DBC) + \underbrace{\triangle\, DCE}_{\triangle DCF} = \mathcal{F}(DBCF)$$

Thus, FD is the desired line.

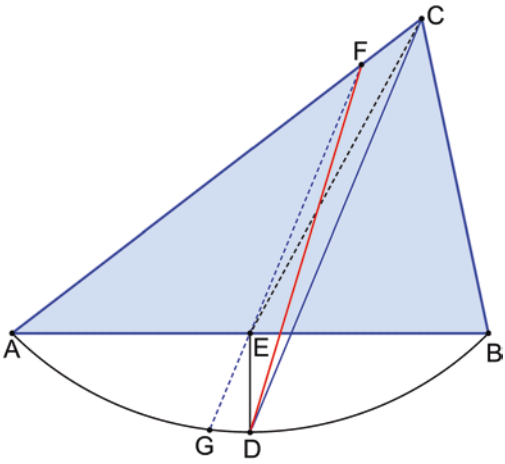


Fig. 10.28 Figure for Theorem 28

Theorem 29
To find the area that contains the fraction p of the circle's area, if the circle's area is to be divided by two parallel chords.

Construction (Fig. 10.29): In the given circle (center M), an isosceles triangle is inscribed with the base $|AB|$ so that the apex lies at the center M and the angle at the center is $\mu = p \cdot 360^{\circ}$. The triangle's symmetry axis intersects the circle at point D. The parallel to AB through M intersects the circle at point C; the parallel to AC through D intersects the circle at point G. The area between the parallel chords AC and DG contains the desired share of the circle's area.

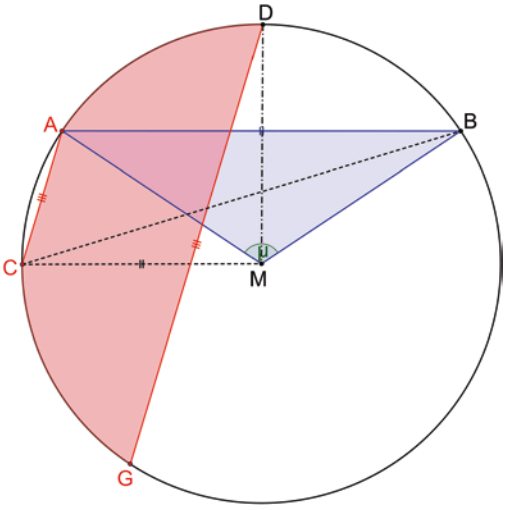


Fig. 10.29 Figure for Theorem 29

Proof The circle sector *AMBD*, consisting of $\triangle AMB$ and the segment *ABD*, has the desired circle area fraction after construction of the angle μ. The $\triangle ACB$ is equal in area to $\triangle ABM$, since it has the same base and height. Thus, the circular sector *AMBD* is equal to the area of $\triangle ACM$ and the segment *ABD*. If you add the segment *AC* to this, you get the segment *BC*. shows The segment *BC* is rotated now by the angle $\frac{\mu}{2}$ into segment GD. If you remove the added segment *AC* again, you get the area between the parallels *AC* and *GD*.

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The Classical Problems of Greek Mathematics

11

11.1 Incommensurability

The fact that the ratio of the diagonal to the side in a square is incommensurable was a great disappointment to the early Pythagoreans, who believed that *(everything is a number)* all ratios are commensurable, i.e. representable by a ratio of whole numbers.

That not all learners know about incommensurability was, for Plato, ridiculous and shameful. In his work [Nomoi 819e] he has the Athenian say:

> And I had to wonder about this state of affairs among us to the highest degree. It seemed to me that this was not possible for humans, but rather only for livestock. And I was ashamed, not only for myself, but for all Greeks.

For Aristotle, incommensurability [Metaphysics 983a] is one of the many examples he draws from mathematics:

> People are first amazed and wonder if things are really as they seem, like the amazement at self-moving puppets, at the solstice, or at the incommensurability of the diagonal (for it seems amazing to everyone that there is something that cannot be measured by the smallest unit). […] No mathematician would be more amazed than if the diagonal were commensurable.

The proof can be found in [Euclid X, 10]:

Given a square of side length $s = |AB|$ and the diagonal $d = |AC|$. We now make the assumption that the ratio $\frac{d}{s}$ is rational (and therefore commensurable) and has no common measure; that is, it holds $gcd(d, s) = 1$.

$$\frac{|AC|}{|AB|} = \frac{d}{s} \Rightarrow \frac{|AC|^2}{|AB|^2} = \frac{d^2}{s^2}$$

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_11

Then according to Pythagoras

$$|AC|^2 = 2|AB|^2 \Rightarrow d^2 = 2s^2$$

With the notation $a|b$ for *a is a divisor of b* it follows

$$d^2 = 2s^2 \Rightarrow 2|d \Rightarrow 4|d^2 \Rightarrow 4\big|s^2 \Rightarrow 2\big|s$$

This is a contradiction to the assumption that the lengths d, s have no common measure. Thus, the ratio $\frac{d}{s}$ cannot be rational; in modern terms $\sqrt{2}$ is irrational or incommensurable. For Aristotle this is the prototype of a contradiction proof *(reductio ad absurdum)*.

In addition to (linear) commensurability, Euclid also defines a square. The basic definitions and theorems about commensurable lines are at the beginning of Book X, like

- Def. (X, 1): Sizes that are measured by the same measure are called commensurable, and incommensurable sizes are those for which there is no common measure.
- Def. (X, 2): Lines are quadratically commensurable if the squares above them are measured by the same area.
- (X, 6): If two sizes have a ratio like a number to a number, then the sizes must be commensurable.
- (X, 7): The squares over linearly commensurable lines have a ratio like a square number to a square number.

A detailed discussion can be found in Kurt von Fritz[1].

11.2 The Constructability According to Euclid

In the *elements*, the conditions for the constructibility according to Euclid of a geometric figure are not given explicitly. According to the postulates in Book I, it must be possible

a) to draw a line from any point to any point
b) to extend a given line arbitrarily
c) to draw a circle with a given point as its center and passing through another given point

It should also be allowed to bring

e) two lines or two circles into intersection
f) to bring a line and a circle into intersection

[1] Fritz K.: The discovery of incommensurability, in the anthology Becker, pp. 271–307.

g) to apply a line segment at one point [I, 2]
h) to erect an equilateral triangle upon a line segment [I, 1]
i) to apply a line segment equal to a smaller line segment on a larger line segment [I, 3]

In Plato the terms circle and line are also used, in Philebos [51C] he writes:

> As beauty of figures I do not mean what the majority would probably assume, such as that of living beings or paintings, but I understand by it […] straight line and circle and the plane and solid [figures] which arise from the circle and the straight line and the angle.

In general, all constructions with a compass (without fixing the circle opening) and a ruler (without markings) are allowed. Lines can only be transferred using g) or i). Other possible constructions are

- to draw a circle through three different points [V, 5]
- to drop a perpendicular from a point outside onto a line [I, 12]
- to erect a perpendicular at a point on a line [I, 11]
- to bisect a line segment or an angle [I, 9][I, 10]
- to draw a tangent to a circle from a point outside [III, 17]

Not included are the so-called *Neusis* constructions, which require a ruler with markings. With these, it would be possible, for example, to move a ruler until a certain marking fits into a given angular field.

All constructible points, lines, lines and circles can be represented in the so-called *Euclidean plane* by coordinates. In this plane (with zero point and unit) lines can be added, subtracted, multiplied and divided arbitrarily. To every line segment of length a alsothe length $\sqrt{a}$ can be constructed (Fig. 11.1).

By linking the specified operations, lines like $x = a \pm \sqrt{b}$ can also be generated; algebraically, x is the root of the quadratic equation

$$x = a \pm \sqrt{b} \Rightarrow x^2 - 2ax + \left(a^2 - b\right) = 0$$

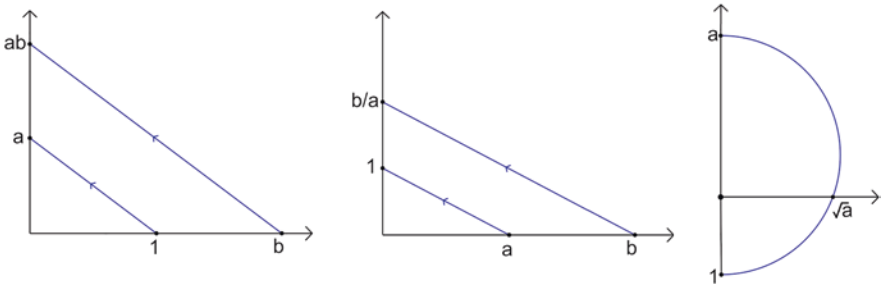

Fig. 11.1 Multiplication, division and square root of lines

Euclid classifies the following irrationalities in Book X of the *Elements*:

$$a+\sqrt{b} \;\therefore\; \sqrt{a}+\sqrt{b} \quad \textit{Binomiale}$$
$$a-\sqrt{b} \;\therefore\; \sqrt{a}-\sqrt{b} \quad \textit{Apotome}$$
$$\sqrt{\sqrt{a}\sqrt{b}} \quad \textit{Mediale}$$

More complicated lines $x=\sqrt{\sqrt{a}\pm\sqrt{b}}$ can also be constructed. The classification of such quadratic irrationalities is the content of Euclid's book X, which because of its difficulty level was called by Simon Stevin *Le croix des mathématiciens* (= Cross of the Mathematicians). Many of Euclid's explanations can be summarized using the following formula (for $a^2 > b$), as already noted by M. Chasles in his History of Geometry (1837)

$$\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

For $a=6,\ b=20$, for example, one obtains a simplification of the square root term

$$\sqrt{6-2\sqrt{5}}=\sqrt{\frac{6+\sqrt{36-20}}{2}}-\sqrt{\frac{6-\sqrt{36-20}}{2}}=\sqrt{5}-1$$

Such apotomes or binomials usually satisfy a biquadratic equation that can be transformed into a quadratic equation using a suitable substitution.

$$x=\sqrt{a\pm\sqrt{b}}\Rightarrow x^4-2ax^2+\left(a^2-b\right)=0\underset{x^2\to y}{\Longrightarrow}y^2-2ay+\left(a^2-b\right)=0$$

A line segment x is therefore constructible if and only if its length satisfies a quadratic or biquadratic equation. Similarly, the angle φ is constructible if the length $\cos\varphi$ is constructible.

A detailed discussion of this is provided by A.D. Steele[2].

11.3 Angle Trisection

Certain angles, such as right angles, can be divided exactly into thirds. Here it is now examined whether any arbitrary angle φ can be trisected according to Euclid. To do this, the angle must be related to its third; this makes the trigonometric formula

[2] Steele A.D.: On the role of the compass and ruler in Greek mathematics, in the collected volume Becker, pp. 146–202.

$$\cos\varphi = 4\left(\cos\frac{\varphi}{3}\right)^3 - 3\cos\frac{\varphi}{3}$$

If the cosine of the given angle is equal to a, then with the substitution $x = \cos\frac{\varphi}{3}$ the equation

$$4x^3 - 3x - a = 0$$

To show that the division is not possible for all angles, it is enough to give *one* such angle. We choose $\varphi = 60° \Rightarrow \cos\varphi = a = \frac{1}{2}$. So it is checked whether the angle 20° can be constructed. Substitution of a gives

$$8x^3 - 6x - 1 = 0$$

To normalize the equation, we substitute $z = 2x$

$$z^3 - 3z = 1$$

It is to be examined whether this polynomial can be factored into a product with a quadratic polynomial with rational coefficients $\in$ Q. The numbers $z = \pm 1$ are not solutions, as one can easily see. It is therefore to be checked whether there is a rational solution $\frac{a}{b}$ with $gcd(a, b) = 1$. Substituting and simplifying shows

$$\left(\frac{a}{b}\right)^3 - \frac{3a}{b} - 1 = 0 \Rightarrow a^3 - 3ab^2 - b^3 = 0$$

Factoring out a shows because of being relatively prime

$$a\left(a^2 - 3b^2\right) = b^3 \Rightarrow a|b^3 \Rightarrow a|b \Rightarrow a = \pm 1$$

Analogously, factoring out b yields

$$b\left(b^2 + 3ab\right) = a^3 \Rightarrow b|a^3 \Rightarrow b|a \Rightarrow b = \pm 1$$

Since a and b are units, it follows that $z = \pm 1$, which has already been excluded. This shows that there is no rational solution. The 60° angle, and thus any angle, can not be trisected according to Euclid.

11.3.1 Constructions for angle trisection

1. Trisection according to Archimedes

The following construction (Fig. 11.2), which Arabic sources attribute to Archimedes, can be found as Theorem 8 in *The Book of Lemmata.* The book is preserved in the work *On the Measurement of the Plane and of the Figures* by Banū Mūsā.

The angle to be trisected, $\measuredangle AMD$, is drawn in any circle (center M). Using a so-called *Neusis* operation, the chord AB is produced until an intersection point C with the straight line DM is obtained and it holds ($|BC| = r$)where r is the radius

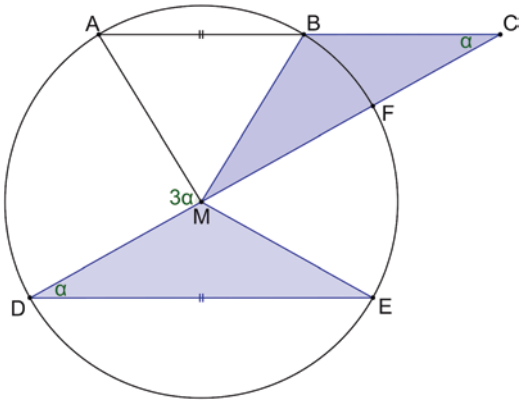

Fig. 11.2 Angle trisection according to Archimedes

of the circle. *MC* cuts the circle at point *F*. Through *D* is drawn the parallel *DE* to the given line *AB*.

Claim: $\measuredangle BME = 3\measuredangle MDE$

$\triangle BMC$ is isosceles because $|BM| = |BC| = r$; base angles be α. $\measuredangle MDE$ is congruent to α being an alternate angle at parallels $AB \parallel DE$ is $\cdot$. The angle *FME* at the center has the same chord *EF* as angle $\measuredangle MDE$ *at the circumference*, it is therefore twice as large: $\measuredangle FME = 2\alpha$. In total, it follows that $\measuredangle BME = 3\alpha$. Since the circle arcs *EB* and *AD* are congruent between parallels, it follows that $\measuredangle BME = \measuredangle AMD$. This proves the claim.

2. During the tradition, the construction was simplified as follows (Fig. 11.3):

The angle to be trisected is drawn as $\measuredangle AMB$ in any circle (center *M*). The radius *AM* is extended beyond *M*. The straight line AM is extended to a point D so that

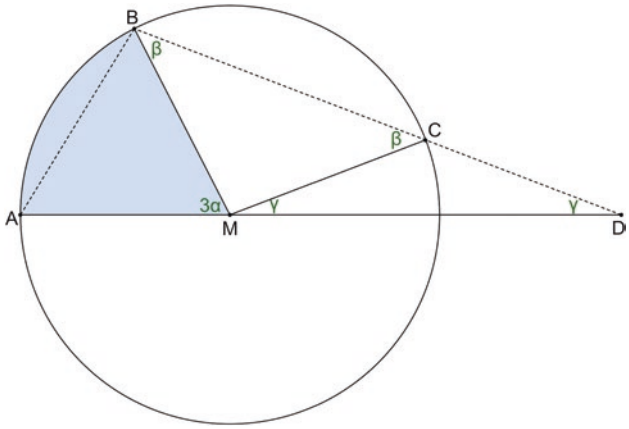

Fig. 11.3 Angle division Archimedes (simplified)

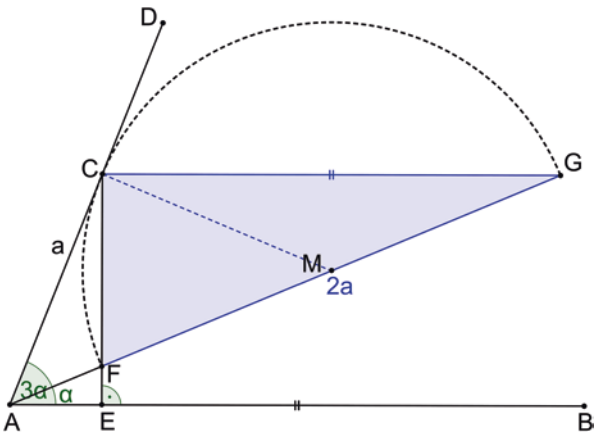

Fig. 11.4 Angle division according to Pappos

the straight line DC intersects the circle at a point C, where $|AM| = |CD| = r \ldots$ Let β be the base angle in $\triangle MCB$ and γ the base angle in $\triangle MDC$. Since $\measuredangle BCM$ is an exterior angle to $\triangle MDC$, it follows that $\beta = 2\gamma$. For the angle $\measuredangle AMD$ is straight, this results in

$$3\alpha + (\pi - \underbrace{2\beta}_{4\gamma}) + \gamma = \pi \Rightarrow \alpha = \gamma$$

This shows that $\measuredangle CMD$ is one third of $\measuredangle AMB$.

3. The following construction is from Book IV of Pappos (Fig. 11.4). The (acute) angle to be divided is $\measuredangle BAD = 3\alpha$. At point A on AD with |AC|= a the parallel to AB is drawn. · In C the perpendicular is dropped to AB with base point E. With the help of a *Neusis* construction, the distance $|FG| = 2a$ is fitted between the perpendicular and the parallel so that the point A lies on the line FG.

Let M be the center of FG. The angle $\measuredangle ECG$ is a right angle, thus, C lies on the Thales circle over FG.. The chord FC has the angle $\measuredangle FMC = 2\beta$ at the center M and the angle $\measuredangle FGC = \beta$ at the circumference. The alternate angle $\measuredangle FGC$is congruent to $\measuredangle BAG$, it follows that $\alpha = \beta$. $\measuredangle BAG$ is thus one third of $\measuredangle BAC$.

4. According to Proclus' report, Hippias of Elis constructed a curve, *Quadratrix* or Trisectrix called, to trisect an angle. The Quadratrix is a curve that is generated in the following way:

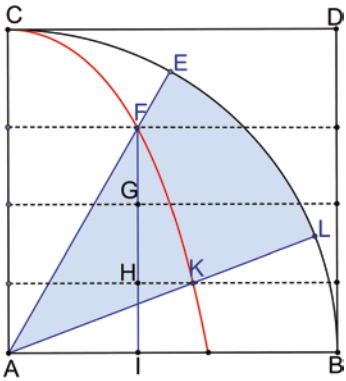

Fig. 11.5 Angular division with Quadratrix

Line *CD* moves uniformly in square *ABCD* downwards. At the same time, line *AC* rotates around point *A* uniformly and synchronously with the displacement of *DC* on *AB*.

The Quadratrix is the set of all intersection points that arise from this simultaneous displacement or rotation (Fig. 11.5).

Given angle $\measuredangle EAB$. The line *AE* intersects the Quadratrix in point *F*. The perpendicular dropped from point *F* upon *AB* gives the foot point I. The line segment FI is risected; one point of division is *H*. The parallel through *H* to *AB* intersects the Quadratrix in point *K*. The line *AK* intersects the circle in point *L*. The angle $\measuredangle LAB$ is the sought third of $\measuredangle EAB$.

Another curve for the trisection of an angle is the Konchoïde of Nikomedes. Reference is made here to the literature (cf. Thomas Sonar[3]).

Disgression to the quadrature of the circle

The (Cartesian) equation of the quadratrix in the unit square is

$$y = x \tan\left(\frac{\pi}{2} y\right)$$

M is the intersection of the Quatratrix with *AB*. The function term is not defined at point *M*. The limit value (with free *x*) delivers

$$\lim_{y \to 0} \frac{y}{\tan\left(\frac{\pi}{2} y\right)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

Thus it applies $|AM| = \frac{2}{\pi}$. From this, a distance of the length $\frac{\pi}{2}$ can be constructed by means of similar triangles. A rectangle with the sides 2 and $\frac{\pi}{2}$ has the area π and is thus equal in area to the unit circle. This is not a construction in the sense of Euclid.

[3] Sonar T.: 3000 Jahre Analysis: Geschichte, Kulturen, Menschen, Springer 2011, pp. 43–45.

11.4 Squaring the Circle

Squaring the circle is the task of constructing a square with the same area as a given circle by using a compass and straightedge (ruler). Since this was not possible in antiquity, the saying *to attempt to square the circle* became synonymous with *attempting to do the impossible.* Since the poet Aristophanes used this saying in his comedy *The Birds*, it can be assumed that the saying was well known at the time.

Since there are polygons whose perimeter is smaller or larger than the inscribed or circumscribed circle, some earlier mathematicians expected that there would also be a polygon with exactly the same circumference as the circle. The first attempts at squaring were made by Hippocrates of Chios (see Chap. 5). Since the unit circle has the area π, a square with the same area would have the area $a^2 = \pi \Rightarrow a = \sqrt{\pi}$. Already Archimedes recognized the impossibility of this representation and, in painstaking calculations, attempted π to enclose it with an (as accurate as possible) upper and lower limit.

11.5 Doubling the Cube

The problem is allegedly based on the demand that a cube-shaped grave, which was considered too small, should receive double the volume. This is what Eratosthenes reports in a letter to Ptolemy II, which was handed down by Eutokios in his commentary on Archimedes. Another version of the story is also told, in which the people of Delos were supposed to double the cube-shaped altar at the request of the oracle and sent envoys to Plato to find a solution. Both stories are probably legends; perhaps they were invented to motivate a mathematical problem with an everyday problem.

If we start from a unit cube, the edge length of the cube of double volume is equal to $a = \sqrt[3]{2}$; a is therefore the root of the cubic polynomial $x^3 - 2$. Since the function $x \to f(x) = x^3 - 2$ is strictly monotonically increasing on R, it only has the one zero $\sqrt[3]{2}$. a cannot be the root of a quadratic polynomial and is therefore not constructible according to Euclid.

Therefore, a solution was sought using a *double proportion*, which is attributed to different authors, such as Hippocrates or Eratosthenes. It follows

$$a : x = x : y = y : b \Rightarrow x^3 = a^2 b$$

For $b = 2a$ it actually follows

$$x^3 = 2a^3 \Rightarrow x = a\sqrt[3]{2}$$

Another solution using the conic section determines x as the intersection of the equilateral hyperbola $xy = \sqrt{2}$ and the parabola $y^2 = x$ (for the unit cube). The construction of Heron for doubling the cube is also known; it is described in Sect. 17.4.

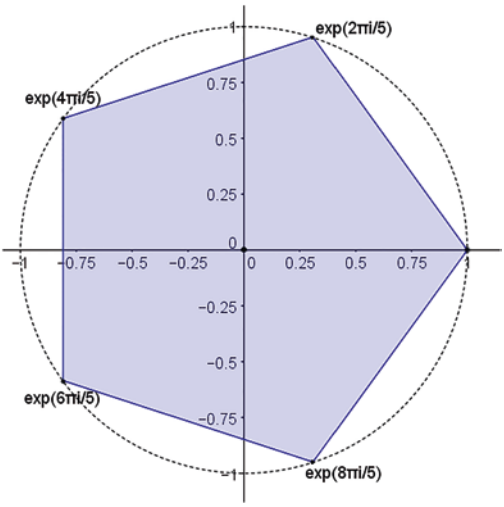

Abb. 11.6 Regular pentagon in the complex number plane

11.6 Constructability of the Pentagon

The regular pentagon

The following study of the construction of a regular pentagon or heptagon is by Pierre Wantzel (1837). If you inscribe a regular pentagon on the unit circle (according to Fig. 11.6), the vertices are given as complex numbers by

$$\zeta_k = e^{2\pi ki/5}; (0 \leq k \leq 4)$$

Since these unit roots have the magnitude 1, they satisfy the circle division polynomial

$$\zeta^5 - 1 = 0$$

An integer root of the polynomial is $\zeta = 1$. Splitting off this root according to the formula of the geometric series yields the polynomial

$$\frac{\zeta^5 - 1}{\zeta - 1} = \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0 \tag{11.1}$$

For the unit root $\zeta = e^{2\pi i/5}$ the following substitution is made

$$x = \zeta + \zeta^{-1} = \zeta + \zeta^4 = e^{i\frac{2\pi}{5}} + e^{i\frac{8\pi}{5}} = 2Re\left(e^{i\frac{2\pi}{5}}\right) = 2\cos\frac{2\pi}{5} \tag{11.2}$$

$$x^2 = \left(\zeta + \zeta^{-1}\right)^2 = \zeta^2 + \zeta^{-2} + 2\zeta\zeta^{-1} = \zeta^2 + \zeta^{-2} + 2 = \zeta^2 + \zeta^3 + 2 \tag{11.3}$$

Re() here denotes the *real part* of the complex number. For polynomial Eq. (11.1) the following division by ζ^2

$$\zeta^2 + \zeta + 1 + \zeta^{-1} + \zeta^{-2} = \left(\zeta^2 + \zeta^{-2}\right) + \left(\zeta + \zeta^{-1}\right) + 1 = 0$$

With the substitutions given above, this yields

$$x^2 + x - 1 = 0$$

This quadratic equation has the solution ·

$x = \frac{1}{2}\left(-1 + \sqrt{5}\right); x > 0$. By expanding with ζ it follows from Eq. (11.2)

$$x = \zeta + \zeta^{-1} = \frac{1}{2}\left(-1 + \sqrt{5}\right) \Rightarrow \zeta^2 - \frac{1}{2}\left(-1 + \sqrt{5}\right)\zeta + 1 = 0$$

The positive solution of this equation yields the fifth unit root

$$\zeta = \frac{1}{4}\left(-1 + \sqrt{5}\right) + \sqrt{\left(\frac{-1 + \sqrt{5}}{4}\right)^2 - 1} \Rightarrow \zeta = \frac{1}{4}\left(-1 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}\right)$$

Since a root of the equation $\zeta^5 - 1 = 0$ is a solution of a quadratic equation, the regular pentagon is constructible in the sense of Euclid.

11.7 Constructibility of the Heptagon

The regular heptagon
The cyclotomic equation is here

$$\zeta^7 - 1 = 0$$

An integer root of the polynomial is $\zeta = 1$. Splitting off this root gives the polynomial

$$\frac{\zeta^7 - 1}{\zeta - 1} = \zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$$

Division by ζ^3 shows

$$\zeta^3 + \zeta^2 + \zeta + 1 + \zeta^{-1} + \zeta^{-2} + \zeta^{-3} = \left(\zeta^3 + \zeta^{-3}\right) + \left(\zeta^2 + \zeta^{-2}\right) + \left(\zeta + \zeta^{-1}\right) + 1 = 0 \tag{11.4}$$

For the seventh unit root $\zeta = e^{2\pi i/7}$ we introduce the substitution

$$x = \zeta + \zeta^{-1} = \zeta + \zeta^6 = 2Re\left(e^{i\frac{2\pi}{7}}\right) = 2\cos\frac{2\pi}{7}$$

$$x^2 = \left(\zeta + \zeta^{-1}\right)^2 = \zeta^2 + \zeta^{-2} + 2\zeta\zeta^{-1} = \zeta^2 + \zeta^{-2} + 2$$

$$x^3 = \left(\zeta + \zeta^{-1}\right)^3 = \zeta^3 + \zeta^{-3} + 3\zeta\zeta^{-2} + 3\zeta^2\zeta^{-1} = \zeta^3 + \zeta^{-3} + 3\zeta^{-1} + 3\zeta \tag{11.5}$$

Inserting into Eq. (11.4) after substitution gives the cubic equation

$$x^3 + x^2 - 2x - 1 = 0 \tag{11.6}$$

The regular heptagon is therefore only constructible if this cubic equation can be decomposed into a product of a quadratic and linear term. As can be easily checked, $x = \pm 1$ is not an integer root. To show that Eq. (11.6) has no rational solution $x = \frac{a}{b}$, the fraction may be assumed to be reduced: $gcd(a, b) = 1$. Inserting and expanding with b^3 gives

$$\left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right)^2 + \frac{2a}{b} - 1 = 0 \Rightarrow a^3 + a^2 b + 2ab^2 = b^3 \Rightarrow a\left(a^2 + ab - 2b^2\right) = b^3$$

This is equivalent to $a|b^3 \Rightarrow a|b$. Because of the lack of common factors, a can only be one unit: $a = \pm 1$. Resolving according to a^3 shows analogously

$$a^3 = b^3 - a^2 b - 2ab^2 = b\left(b^2 - a^2 - 2ab\right)$$

This results in $b|a^3 \Rightarrow b|a \Rightarrow b = \pm 1$. In total, only $x = \frac{a}{b} = \pm 1$ can be. However, we have already excluded this. So Eq. (11.6) is not rationally solvable; $x = \cos\frac{2\pi}{7}$ and thus the regular heptagon is not constructible according to Euclid.

Outlook

C.F. Gauß could already show as a first semester student (1796) that the regular n-gon is constructible if and only if n has the factorization

$$n = 2^k p_1 p_2 \cdots p_m; \quad k, m \in \mathbb{N}_0$$

The factors p_i have to be Fermat primes of the form $2^{2^j} + 1$. In the simplest case for $n = 2^{2^2} + 1 = 17$ a prime number results. Thus the regular 17-gon is constructible. Gauß could factorize the corresponding cyclotomic polynomial

$$\frac{\zeta^{17} - 1}{\zeta - 1} = \sum_{i=0}^{16} \zeta^i$$

He found

$$\cos\frac{2\pi}{17} = \frac{1}{16}\left(-1 + \sqrt{17}\right) + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

A *Neusis*-construction of the heptagon of Archimedes is shown in Sect. 12.4.

11.8 Quadratures of Lunes

The area of the lune $(AECF)$ can be calculated from the area decomposition (Fig. 11.7):

$$lune(AECF) + sector(ABCE) = kite(ABCD) + sector(ADCF)$$

Given is a circle with center D and radius $r = |DC|$, another one with center B and radius $R = |BE|$. For the area of the sector $(ABCE)$ the following applies in radian measure

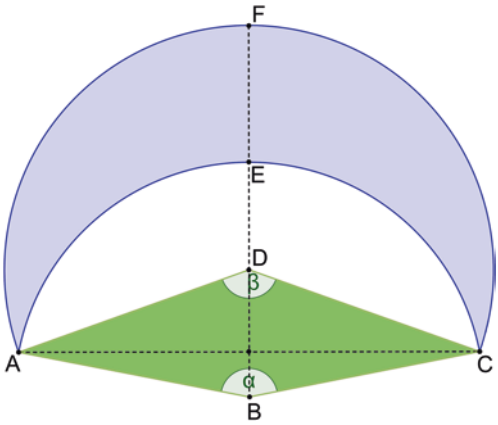

Fig. 11.7 Figure for the quadrature of the lune

$$sector(ABCE) = \alpha R^2$$

Analogously, for sector $(ADCF)$

$$sector(ADCF) = \beta r^2$$

The area of the kite $ABCD$ consists of two pairs of congruent right-angled triangles. Thus, according to the triangle formula $A = \frac{1}{2}ab\sin\gamma$

$$kite(ABCD) = Rr\sin|\beta - \alpha|$$

Thus

$$lune(AECF) = \beta r^2 - \alpha R^2 + Rr\sin(\beta - \alpha)$$

The lune can be squared if

$$lune(AECF) = kite(ABCD) \Rightarrow \beta r^2 - \alpha R^2 = 0 \Rightarrow \frac{r}{R} = \sqrt{\frac{\alpha}{\beta}} \quad (*)$$

These radii are linked to each other in the triangle BCD by the sine theorem

$$\frac{r}{\sin\alpha} = \frac{R}{\sin(\pi - \beta)} \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{r}{R} \Rightarrow \frac{\sin\alpha}{\sin\beta} = \sqrt{\frac{\alpha}{\beta}}$$

It is to be checked whether the above formula has a solution. It can be shown that there are only finitely many solutions here, namely 5 cases $\frac{\beta}{\alpha} \in \{2, 3, \frac{3}{2}, 5, \frac{5}{3}\}$. In these cases, a solution representable by a square root results.

Case (1): Here $\beta = 2\alpha$. Therefore, it follows that

$$\frac{\sin 2\alpha}{\sin\alpha} = \sqrt{2} \Rightarrow \frac{2\sin\alpha\cdot\cos\alpha}{\sin\alpha} = \sqrt{2} \Rightarrow \cos\alpha = \frac{1}{2}\sqrt{2} \Rightarrow \alpha = 45°,\ \beta = 90°$$

Case (2): Here $\beta = 3\alpha$. Therefore, it follows that

$$\frac{\sin 3\alpha}{\sin\alpha} = \sqrt{3} \Rightarrow \frac{3\sin\alpha - 4(\sin\alpha)^3}{\sin\alpha} = \sqrt{3} \Rightarrow 3 - 4(\sin\alpha)^2 = \sqrt{3}$$

The quadratic equation has a solution

$$\sin\alpha = \frac{1}{2}\sqrt{3-\sqrt{3}} \Rightarrow \alpha = 34.265^\circ,\ \beta = 3\alpha$$

Case (3): Here $\beta = \frac{3}{2}\alpha$. The ansatz $\alpha = 2\varphi,\ \beta = 3\varphi$ yields

$$\frac{\sin 3\varphi}{\sin 2\varphi} = \sqrt{\frac{3}{2}} \Rightarrow \frac{3\sin\varphi - 4(\sin\varphi)^3}{2\sin\varphi\cos\varphi} = \sqrt{\frac{3}{2}} \Rightarrow \frac{3-4(\sin\varphi)^2}{2\cos\varphi} = \sqrt{\frac{3}{2}}$$

The substitution $(\sin\varphi)^2 = 1-(\cos\varphi)^2$ shows

$$\frac{3-4\left[1-(\cos\varphi)^2\right]}{2\cos\varphi} = \sqrt{\frac{3}{2}} \Rightarrow -1+4(\cos\varphi)^2 = 2\sqrt{\frac{3}{2}}\cos\varphi$$

With the substitution $x = \cos\varphi$, the quadratic equation with a solution

$$\Rightarrow 4x^2 - \sqrt{6}x - 1 = 0 \Rightarrow x = \cos\varphi = \frac{\sqrt{6}+\sqrt{22}}{8}$$

Here, $\varphi = 26.8124^\circ\ldots, \alpha = 2\varphi,\ \beta = 3\varphi$.

Case (4): Here $\beta = 5\alpha$. It follows that

$$\frac{\sin 5\alpha}{\sin\alpha} = \sqrt{5} \Rightarrow \frac{(\sin\alpha)^5 + 5\sin\alpha(\cos\alpha)^4 - 10(\sin\alpha)^3(\cos\alpha)^2}{\sin\alpha} = \sqrt{5}$$

Simplifying shows

$$(\sin\alpha)^4 + 5(\cos\alpha)^4 - 10(\sin\alpha)^2(\cos\alpha)^2 = \sqrt{5}$$

With the substitution $(\cos\alpha)^2 = 1-(\sin\alpha)^2$, the biquadratic equation

$$16(\sin\alpha)^4 - 20(\sin\alpha)^2 = \sqrt{5}-5$$

With the substitution $x = (\sin\alpha)^2$, the equation with a solution

$$16x^2 - 20x + \left(5-\sqrt{5}\right) = 0 \Rightarrow (\sin\alpha)^2 = \frac{5-\sqrt{5+4\sqrt{5}}}{8}$$

$$\Rightarrow \sin\alpha = \frac{1}{2}\sqrt{\frac{5-\sqrt{5+4\sqrt{5}}}{2}}$$

Here follows $\alpha = 23.4391^\circ\ldots, \beta = 5\alpha$.

Case (5): Here applies $\beta = \frac{5}{3}\alpha$. The approach $\alpha = 3\varphi,\ \beta = 5\varphi$ delivers

$$\frac{\sin 5\varphi}{\sin 3\varphi} = \sqrt{\frac{5}{3}} \Rightarrow \frac{(\sin\varphi)^5 + 5\sin\varphi(\cos\varphi)^4 - 10(\sin\varphi)^3(\cos\varphi)^2}{3\sin\varphi - 4(\sin\varphi)^3} = \sqrt{\frac{5}{3}}$$

$$\Rightarrow \frac{(\sin\varphi)^4 + 5(\cos\varphi)^4 - 10(\sin\varphi)^2(\cos\varphi)^2}{3-4(\sin\varphi)^2} = \sqrt{\frac{5}{3}}$$

With the substitution $(\cos\varphi)^2 = 1 - (\sin\varphi)^2$ follows with $x = \sin\varphi$

$$\frac{16(\sin\varphi)^4 - 20(\sin\varphi)^2 + 5}{3 - 4(\sin\varphi)^2} = \sqrt{\frac{5}{3}} \Rightarrow 12x^4 + \left(\sqrt{15} - 5\right)x^2 = \frac{3\sqrt{15} - 15}{4}$$

This is a biquadratic equation, which can be converted into a quadratic equation by means of substitution. One of the solutions is

$$x = \sin\varphi = \frac{1}{2}\sqrt{\frac{15 - \sqrt{15} - \sqrt{60 + 6\sqrt{15}}}{6}} = 0.28893\ldots$$

Here results $\varphi = 16.7939\ldots^\circ \Rightarrow \alpha = 50.3818\ldots^\circ, \beta = 83.970\ldots^\circ$

Cases 1 to 3 lead to constructions that were already found by Hippocrates. Cases 4 and 5 were only discovered in 1840 by the German mathematician T. Clausen[4] after preliminary work by Vieta and Euler.

11.9 The Division in extreme and mean ratio (EMR)

Geometry holds two great treasures: one is the Pythagorean theorem, the other is the Golden Section.

We can compare the first to a bushel of gold, the second we can call a precious jewel. (J. Kepler).

Euclid calls the division according to *the extreme and mean ratio;* it can be derived from the transformation of a square into an a rectangle of equal area [Euclid II, 11]. Pacioli and Kepler use the term *sectio divina* (divine section). The term *Golden Section* only arises in the nineteenth century.

Construction according to Euclid

Given is the line $|AB| = a$. It is to be divided by a point D with $|AD| = x$ such that

$$\frac{a}{x} = \frac{x}{a - x} \Leftrightarrow a(a - x) = x^2$$

(Fig. 11.8). Above the line AB the square $ABCI$ is constructed; the point E is the midpoint of AC. The circle with center E and radius $|EB|$ intersects the extension of AC at the point F. Now the square is constructed over AF; the side length is $x = |AF| = |AD|$. The circle with center B and radius $|DH|$ intersects the square side in I. The rectangle $DHIB$ is, according to Euclid, equal in area to the

[4] Clausen T.: Vier neue mondförmige Flächen, Journal für die reine und angewandte Mathematik 21(1840), pp. 375–376.

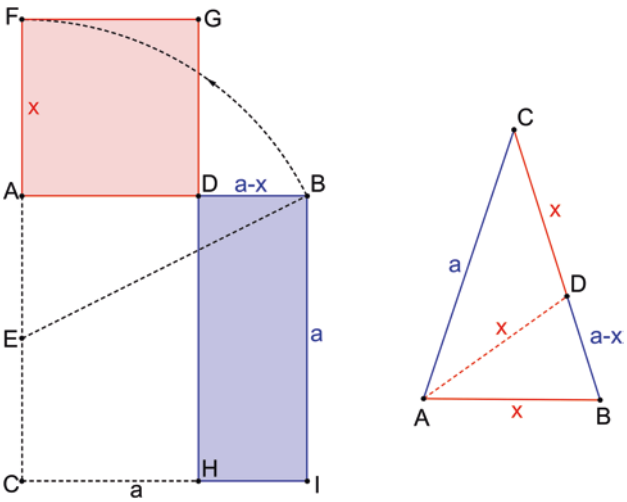

Fig. 11.8 Continuous division according to Euclid

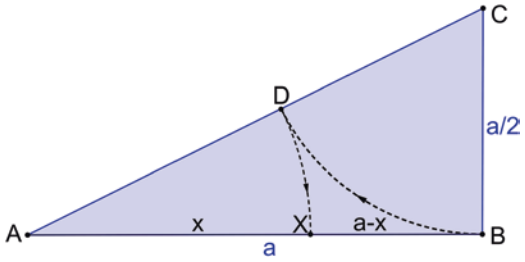

Fig. 11.9 Division in EMR according to Heron

square *ADGF*. The construction of the isosceles triangle, whose legs are divided according to EMR, can also be found in [Euclid IV, 10]. It has the special property that a base angle is twice as large as the angle at the top.

Construction according to Heron

Something simpler is the construction used by Heron (Fig. 11.9). Here $\triangle ABC$ is right-angled with the sides $|BC| = \frac{1}{2}|AB|$. A circle with centre C and radius $|BC|$ intersects the hypotenuse AC at the point D. A circle with centre A and radius $|AD|$ gives as the point of intersection X with AB the desired point of division.

Euclid treats the division in EMR twice in the *elements*, namely in the construction of the triangle, in which a base angle is twice the angle at the top (Euclid IV, 10) and in the construction of the regular decagon (Euclid XIII, 10).

Euclid IV, 10

It is possible to construct an isosceles triangle in which one base angle is twice as large as the angle at the top (Fig. 11.10).

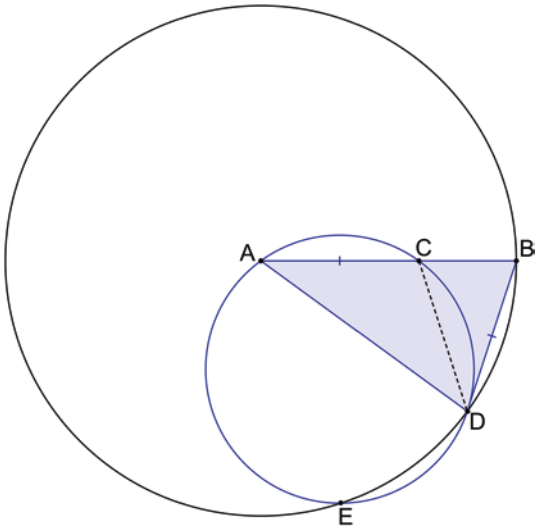

Fig. 11.10 Figure to Euclid IV, 10

Proof: Given is the line AB divided in EMR at the division point C. A circle with center A and radius $|AB|$ is drawn. From point B the line $|BD| = |AC|$ is marked on the circumference. The circumcircle of $\triangle ACD$ intersects the circle at A at point E. $\triangle ACD$ is the desired triangle. Since C divides the line AB in EMR, $|AC|^2 = |AB||BC|$. applies. Because of $|BD| = |AC|$ it follows that $|BD|^2 = |AB||BC|$. According to the secant-tangent theorem, it follows that the line $|BD|$ is a tangent to the circumcircle of $\triangle ACD$; AB is the corresponding, opposite secant. To the chord CD, $\measuredangle CDB$ represents the chord-tangent angle and $\measuredangle DAC$ the corresponding angle at the circumference; both angles are therefore congruent $\measuredangle CDB = \measuredangle DAC$.

If you add to the last equation $\measuredangle CDA$, you get $\measuredangle BDA = \measuredangle DAC + \measuredangle CDA$. The angles of the right side are interior angles of $\triangle ACD$; for the exterior angle it follows that $\measuredangle DCB = \measuredangle DAC + \measuredangle CDA$. This equality shows $\measuredangle BDA = \measuredangle DCB$. $\measuredangle BDA$ is also congruent to $\measuredangle CBD$, since the sides $|AB| = |AD|$ are also radii of the circle with center A. $\measuredangle DBA$ is thus congruent to $\measuredangle BCD$. In total, the angles $\measuredangle BDA$, $\measuredangle BCD$ and $\measuredangle DBA$ are congruent in pairs.

Since the angles $\measuredangle BCD = \measuredangle DBC$ are congruent, the $\triangle ACD$ is isosceles with $|CD| = |AC|$. Since it holds according to assumption: $|BD| = |AC|$ it follows that $|BD| = |CD|$; thus, $\triangle DBC$ is isosceles, too. Finally, $\measuredangle BDA$ or $\measuredangle DBA$ is the double of $\measuredangle BDC$. This is the claim.

Remark: Euclid's proof is here extensive because he cannot use the similarity of the triangles $\triangle ADB$ or $\triangle BCD$. Euclid only uses similar triangles in book VI of the elements. Thus, this proof could already stem from the Pythagoreans. With its help, the regular pentagon can be constructed after all.

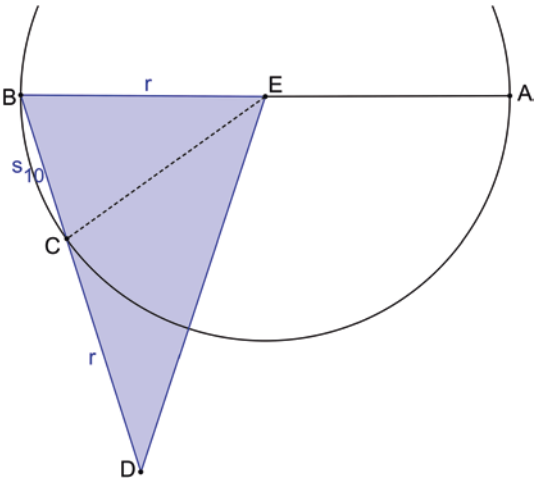

Fig. 11.11 Figure to Euclid XIII, 9

Euclid XIII, 9

This is the lemma to Euclid XIII, 10.

If you add the sides of the hexagon and decagon inscribed in the same circle together, the sum of the line segments will be divided in EMR; the larger section is the hexagon side (Fig. 11.11).

Proof: In the circle with diameter $|AB| = 2r$, the line segment $|BC| = s_{10}$ is the side of the regular decagon inscribed. BC is produced by the line segment $|CD| = s_6 = r$ of the regular hexagon inscribed to the endpoint D. The arc (ACB) is equal to five times the arc (BC), so arc $(AC) = 4$ arc (CB). Since the arcs are proportional to the correspondingangles at the center, it follows that

$$\frac{Bogen(AC)}{Bogen(CB)} = \frac{\sphericalangle AEC}{\sphericalangle ECB}$$

Thus also applies $\sphericalangle AEC = 4\sphericalangle ECB$. Since $\triangle BCE$ is isosceles, $\sphericalangle EBC = \sphericalangle ECB$ applies. It follows $\sphericalangle AEC = 2\sphericalangle ECB$. After construction, $\triangle CDE$ is isosceles, so $\sphericalangle CED = \sphericalangle CDE$, and thus also $\sphericalangle ECB = 2\sphericalangle EDC$. As shown above, $\sphericalangle AEC = 2\sphericalangle ECB$ applies, and thus also $\sphericalangle AEC = 4\sphericalangle EDC$. Also shown is: $\sphericalangle AEC = 4\sphericalangle BEC$, and thus also $\sphericalangle EDC = \sphericalangle BEC$. Since the triangles $\triangle BCE$ and $\triangle BDE$ are similar, it follows

$$\frac{|DB|}{|BE|} = \frac{|EB|}{|CB|}$$

Since $|BD| > |DC|$ also $|DC| > |CB|$ follows. The distance BD is thus divided in EMR and DC is the larger section.

Euclid XIII, 10

If you inscribe a regular pentagon, hexagon, and decagon in the same circle and put the three sides together to form a triangle, you get a right-angled triangle (Fig. 11.12)

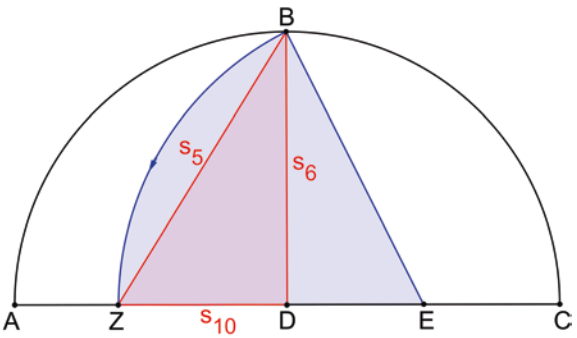

Fig. 11.12 Figure for Euclid XIII, 10

Proof: Let E be the midpoint of DC and at the same time the center of a circle with the radius $|BE|$. Then it follows: $|DE| = |EC|$; $|BE| = |ZE|$. Then, according to [Euclid II, 6],

$$|EZ|^2 = |ED|^2 + |CZ||ZD|$$

It follows from Pythagoras' theorem

$$|EZ|^2 = |EB|^2 = |ED|^2 + |DB|^2 \Rightarrow |DB|^2 = |DG|^2 = |CZ||ZD|$$

The latter is the proportion of the division in EMR

$$\frac{|DZ|}{|DC|} = \frac{|DC|}{|ZC|}$$

$|DG|$ is at the same time the regular hexagon side. According to the auxiliary theorem (XIII, 9), $|DZ|$ is the regular decagon side and thus the chord to the 36° angle. For the second angle, $\measuredangle DBZ = 72°$ follows; $|BZ|$ is therefore the regular pentagon side.

The golden ratio

According to the above definition, a line with sections a, b is called divided in golden ratio, if the following applies

$$\frac{a+b}{a} = \frac{a}{b} \Rightarrow 1 + \frac{b}{a} = \frac{a}{b}$$

The ratio $\frac{a}{b} = \varphi$ is called the number of the Golden Ratio or directly *Golden Ratio.* It applies to the top

$$1 + \frac{1}{\varphi} = \varphi \tag{11.7}$$

Transforming delivers the quadratic equation with the positive solution

$$\varphi^2 = \varphi + 1 \quad (11.8) \Rightarrow \varphi = \frac{1}{2}\left(1 + \sqrt{5}\right) \approx 1.618034$$

Fig. 11.13 Macao stamp on the Golden Section

This means illustratively that the larger part (usually called Major) relates to the smaller part (Minor) like φ. The choice of the letter φ is supposed to go back to the memory of the famous Greek sculptor Phidias (490–432 BC), who was probably the first to realize the principle of the Golden Ratio in architecture. Figure 11.13 shows a postage stamp from Macao, which illustrates the division methods shown here.

According to R. Herz-Fischler[5] The investigation of division in EMR was part of the mathematical research program at the Academy. He constructs the sequence as follows:

1. At the beginning of the Academy (around 386 BC) the first research task was *Regular Polygons*
2. The inscription of the regular pentagon in the circle led to Euclid III, 36–37 or II, 6 and the construction according to (II, 11).
3. A second research program Regular Polyhedra was started; it does not yet require the theory of proportions in Book VI.
4. The theorems Euclid (XIII, 3–5) are developed. These are continued in (XIII, 6a) or (XIII, 7a) with the construction of the icosahedron and the dodecahedron.
5. As a result of these results, the theory of irrational numbers was continued in (XIII, 1–2) and (XIII, 6) or in (XIII, 17).

[5] Herz-Fischler R.: A Mathematical History of the Golden Number, Dover 1998, p. 98.

6. In the classification of the terms entering the icosahedron, the theorems Euclid (XIII, 11) and (XIII, 16) are formulated without using the proportions from Book V.
7. After the completion of the theory of proportions in Book V, Definition 3 from Book VI and (VI, 30) are developed as a replacement for (II, 11).

Further References

Becker, O.: Das mathematische Denken der Antike. Vandenhoek & Ruprecht, Göttingen (1967)
Bold, B.: Famous Problems of Geometry. Dover, New York (1969)
Herz-Fischler, R.: A Mathematical History of the Golden Number, p. 98. Dover, New York (1998)

Archimedes of Syracuse

12

Archimedes (Αρχιμήδης) was born in 287 BC, the son of the astronomer *Pheidias,* in the port city of Syracuse on the island of Sicily. The date of birth can be deduced from a report by the Byzantine historian Johannes *Tzetzes,* who noted that Archimedes (Fig. 12.1) lived to be 75 years old. His date of death is known because it was caused by the Roman conquest of Syracuse during the Second Punic War.

The Second Punic War was fought between 218 and 202/201 BC and is known for Hannibal's crossing of the Alps (with elephants). Here the Romans suffered their worst defeat ever in the Battle of Cannae in 216. But the Romans did not lose the war. To get Sicily back under their control, the Romans sent the experienced general Marcus Marcellus to the island in 214, who had already distinguished himself in 222 against the Gauls in northern Italy. For Syracuse, a two-year siege began by land and sea. Only in 212 was the city taken. During the siege, as Plutarch (Life of Marcellus, 15) writes, the war machines invented by Archimedes are said to have been a great obstacle for the Roman attackers. The reports that Archimedes is said to have set fire to the sails of Roman ships with large burning mirrors are not credible, since the events were physically impossible with the technology of that time. The historian Polybius, on the other hand, reports of a bloodless conquest, Syracuse was overwhelmed because the inhabitants were not ready for battle due to a three-day festival in honor of the goddess Artemis.

Note: It should be mentioned here the experiment of the Greek engineer Iannis Sakkas, which he made in 1973 in the port of Skaramagas, the military area of Piraeus: He placed 70 highly polished copper mirrors of the format 1 m by 0.5 m at a distance of 50 m in front of a plywood model of a Roman ship. After precisely focusing the mirrors, Sakkas actually managed to set the model ship on fire.

Archimedes, the most important scholar of the Hellenistic period, was not formally part of the Alexandrian school. However, it is certain that he Alexandria visited and there came into contact with scholars, as his correspondence with *Konon, Dositheos* and *Eratosthenes* testifies. Konon, who was one generation older, was an astronomer at the court of Ptolemy III. Eugertes (= benefactor) and probably

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_12

Fig. 12.1 Archimedes (Wikimedia Commons)

encouraged Archimedes to deal with purely mathematical problems. After his return to Syracuse, Archimedes corresponded regularly with Konon, which he continued after Konon's death with his student Dositheos. Each of the five surviving letters represents a separate mathematical treatise:

- 1. Quadrature of the Parabola
- 2./3. On the Sphere and the Cylinder
- 4. On Conoids and Sphairoids
- 5. On Spirals.

The mathematical significance of this correspondence is hardly to overestimate. In these letters, Archimedes developed new methods for measuring the area of parabolas, applications of the method of exhaustion of Eudoxos for determining the volume of geometric bodies. Archimedes here develops the method of nested intervals; i.e. the size to be determined is included between two integral sums, the difference of which can be made arbitrarily small. The desired size then appears as the common limit. This allowed him to determine the surface of the sphere, segments of paraboloid and hyperboloid, and the volume of the triaxial ellipsoid.

With the help of the same procedure, he also solved more difficult tasks, such as determining the lengths of arcs and the surface areas of a series of curved surfaces. All of these tasks are contained in the treatises "On the Sphere and Cylinder", "On the Conoids and the Sphairoids" and "On the Spirals". For him, the results were important, for example, that the surface of the sphere is related to the surface area of a large circle as 4: 1 or the volume of a sphere to the volume of

the circumscribed cylinder as 2: 3. He is said to have wished this last figure to be his grave stone.

In addition to the mentioned surface and volume determinations, Archimedes developed a method for determining the tangent of a curve that anticipates the principle of differentiation. In the treatise "On Spirals", he applies the procedure to the spiral named after him $r = a\varphi$. The considerations of Archimedes are generally valid and can be applied to any differentiable curve.

Another topic that occupied Archimedes was the lever; the associated, earlier writings "On the Balance" and "On the Lever" have not been preserved. In his work "On the Equilibrium of Plane Figures" he gave the mathematical theory of equilibrium, followed by a theory of the center of gravity. After stating a number of general propositions, he applies these to the determination of the center of gravity of a triangle, parallelogram, trapezoid, and segment of a parabola.

Some fragments of the lost writings are found in Heron (in the Mechanics) and Pappus (in the *Collectio*); Pappus gives the precise definition of the center of gravity of a body according to Archimedes. His most popular discovery is undoubtedly that of buoyancy. The fairy-tale story of how he determined the gold content of the crown of the tyrant Hieron of Syracuse by means of buoyancy is known from the account of Vitruv (Book IX, Preface 9-12). The essential thing, namely the "Archimedean Principle", does not play a role in the story. This appears in the letter *Ephodos,* addressed to Eratosthenes and found in Byzantium in 1905 by J. L. Heiberg. The letter was found together with a fragment that came from the lost treatise "On Floating Bodies."

Not all of Archimedes' mathematical works have been preserved. Fortunately, the Arabic tradition has preserved the writings "Lemmata", "On the Heptagon" and "On Circles Touching Each Other". Some of the theorems proven by Archimedes can still be found in a treatise by the scholar al-Biruni (973–1048). Some writings were preserved through an early Latin translation, such as *De mensura circuli* by Plato of Tivoli and Gerard of Cremona. The latter also discovered parts of the writing "On the Sphere and Cylinder" in a manuscript "Verba filorum" of the Banū Mūsā brothers[1]; the exact Latin title is: "Verba filorum Moysi Sekir, i.e. Maumeti, Hameti, Hasen".

Plutarch writes about Archimedes' work in *Vitae parallelae* (Life of Marcellus, 17, 4)

> It is not possible to see deeper and more difficult questions treated in the whole geometry which are also explained simply and clearly. Some attribute this success to his genius; others think that it is due to his incredible diligence that everything achieved seems to be achieved without effort and with ease.

Many stamps and coins show an alleged image of Archimedes (Fig. 12.2). As it turned out, this is the bust of King Archidamos III of Sparta (third century BC).

[1] Verba filorum of the Banū Mūsā, in Clagett's anthology p. 223–241.

Fig. 12.2 Stamps with false Archimedes images (Wikimedia Commons)

In the following, excerpts from his works are presented. Here we do not go into the famous cattle problem, which was curiously discovered by the German poet G.E. Lessing. A detailed report is given by Jacques Sesiano[2] .

12.1 On the Centers of Gravity

In his work *De planorum aequilibris* (On the Equilibrium of Planes), Archimedes, after introducing numerous postulates, defines the following centers of gravity

- Theorem (4): If two equal weights do not have the same center of gravity, then the center of gravity is the midpoint of the connecting line.
- Theorem (6+7): Unequal weights are in equilibrium when their weights are inversely proportional to their lever arms.
- Theorem (10): The center of gravity of a parallelogram is the intersection of the diagonals.
- Theorem (13): The center of gravity of a triangle is the intersection of the medians.

In Theorem (15), Archimedes deals with the center of gravity of a trapezoid (Fig. 12.3):

Construction: If $ABCD$ is the given trapezoid, then the midpoints E and F of the parallel sides are determined. By drawing two parallels GH and KL to the base line AB, the legs are divided into thirds. The intersection points of the lines AF and KL as well as GH with CE are M and N. The intersection of the lines MN and EF is the desired center of gravity S.

The trapezoid $ABCD$ can be divided into the $\triangle ABC$ and $\triangle ACD$. Since the centers of gravity divide the center lines in the ratio 2:1, the third division provides

[2] Sesiano J.: An Introduction to the History of Algebra, American Mathematical Society 2009, pp. 27–31.

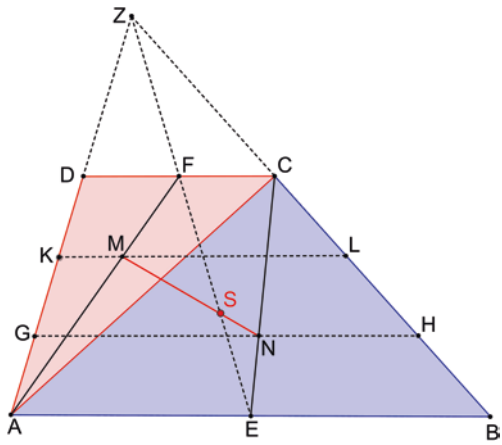


Fig. 12.3 To the trapezoid center of gravity

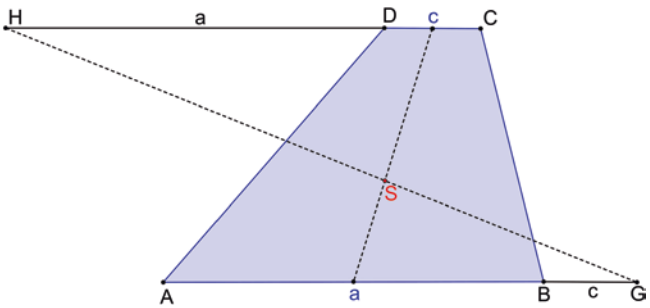


Fig. 12.4 Modern construction of the trapezoidal center of mass

the centers of gravity N of the $\triangle ABC$ and M of the $\triangle ACD$. The union of the triangle areas therefore has the center of gravity on the connecting line MN. For symmetry reasons, the center of gravity must also lie on the connecting line of the midpoints of the parallel sides. Thus, the center of gravity S of the trapezoid $ABCD$ is given by the intersection of the lines MN and EF.

An alternative construction of the center of mass S of the trapezoid $ABCD$ is shown in Fig. 12.4:

12.2 Problem of the Broken Chord

Al-Bīrūnī (973–1050) attributes the following problem to Archimedes in his work *Book of the Determination of Circle Tensions*:

If two chords AB and BC (with $|AB| < |BC|$) are in a circle, the union of the two chords (according to the Arabic source) is called a *broken chord* ABC (Fig. 12.5). If M is the center of the circular arc over $\widehat{AC}$ and F is the foot point of

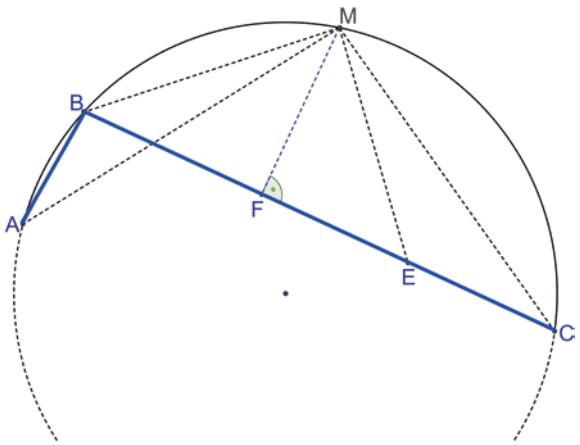


Fig. 12.5 To the problem of the broken chord

the perpendicular from M upon BC, then F is the center of the broken chord and it holds

$$|AB| + |BF| = |FC|$$

Proof: The auxiliary point E is selected on BC such that $|AB| = |EC|$ holds. This is possible because $|BC|>|AB|$ is assumed. Since M is the center of the circular arc over AC, the chords are also congruent with the arc: $|AM| = |CM|$. After construction, $|AB| = |EC|$ holds. $\measuredangle BAM$ or $\measuredangle ECM$ are congruent angles at the circumference over the common chord BM. Thus, the triangles $\triangle ABM$ and $\triangle ECM$ are congruent, since they match in two sides and the included angle.

Since the sides $|BM| = |ME|$ are thus congruent, $\triangle BEM$ is isosceles and the corresponding median MF bisects the base BE. Thus,

$$|AB| = |EC| \ \therefore \ |BF| = |FE|$$

Addition yields the claim

$$|AB| + |BF| = |FE| + |EC| = |FC|$$

Al-Bīrūnī extends the *theorem of the broken chord* to the following theorem

$$|MC|^2 = |AB||BC| + |BM|^2$$

He provides several proofs for this, using the theorem of Ptolemy.

12.3 The Regular Heptagon

In the pursuit of constructing a regular heptagon, Archimedes found the following remarkable *Neusis* construction (Fig. 12.6).

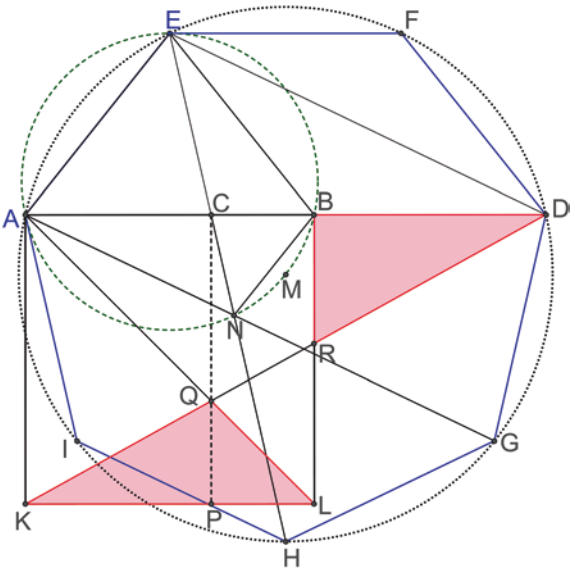


Fig. 12.6 Neusis construction for the regular heptagon

Construction Description

Construct the arbitrary square $ABKL$, draw the diagonal AL, and produce the line segment AB beyond B. Let R be an arbitrary point of division on the line segment BL. The line KR intersects the extension of AB at the point D.

The point R is now chosen so that the triangle KLQ is is equal in area to the triangle BRD!

The intersection of KD with AL is the point Q. The parallel through Q to the side AK intersects with the square at the points C and P. The point E is given by the conditions: $\triangle ACE$ is isosceles with $|AC| = |CE|$ and $\triangle EBD$ is isosceles with $|BE| = |BD|$. E is therefore the intersection of the circle with center C and radius $|AC|$ with the circle with center B and radius $|BD|$ (not shown). The desired side of the regular heptagon is therefore $s_7 = |AE|$.

Another vertex of the heptagon is point D. The circumscribed circle's center point M of the polygon can be found by the intersection of the perpendicular bisectors of AE resp. ED. By applying the side length s_7 on the circumference of the circumscribed circle, the heptagon can be drawn completely. To check the drawing, the triangle's circumscribed circle ABE can be used. The intersection of the line AC with this circumscribed circle provides point N. The vertices G resp. H lie therefore on the line AN resp. EN.

Hint: The corresponding proof is neither to be found with Archimedes, nor in the commentary of the handed down Arabic manuscript. The equality of areas of the triangles KLQ and BRD can be reduced to the validity of $|AB| \cdot |AC| = |BD|^2$. The area of KLQ is $\frac{1}{2}|PQ| \cdot |KL|$, the one of BRD $\frac{1}{2}|BR| \cdot |BD|$. Thus, $|PQ| \cdot |KL| = |BR| \cdot |BD|$ or

$$\frac{|PQ|}{|BD|} = \frac{|BR|}{|KL|}$$

The triangles KPQ and BRD are similar, since they have the right angle and the congruent alternate angles $\measuredangle QKP = \measuredangle BDR$. Thus follows

$$\frac{|PQ|}{|KP|} = \frac{|BR|}{|BD|}$$

From both proportions follows

$$\frac{|BC|}{|DI|} = \frac{|DI|}{|BF|} \Rightarrow |BC| \cdot |BF| = |DI|^2$$

Because of the congruence $|BC| = |AD|$ resp. $|BF| = |AE|$, follows the claim $|AD| \cdot |AE| = |DI|^2$. In the same way it can be shown $|EI| \cdot |ED| = |AE|^2$.

12.4 The Book of Measuring the Circle

The work *De Mensura Circuli*[3] (The *measuring the circle*) by Archimedes is only partially preserved. The three surviving theorems are:

1. The circle is congruent to a right-angled triangle in which one leg is equal to the radius and the other is equal to the circumference of the circle (Fig. 12.7).

$$A = \frac{1}{2} r \cdot 2\pi r = \pi r^2$$

2. The area of the circle A has the ratio of the square of the diameter almost 11:14.

$$\frac{A}{(2r)^2} \approx \frac{11}{14} \Rightarrow A \approx 3\frac{1}{7} r^2$$

3. The circumference of the circle U is three times as large as the diameter and a little bit larger, namely less than $\frac{1}{7}$, but more than $\frac{10}{71}$

$$3\frac{1}{7} \cdot 2r > U > 3\frac{10}{71} \cdot 2r$$

Remark: Theorem (1) has already been strongly criticized in antiquity, inter alia by Pappus and Simplicius, since it is not possible to construct the triangle within the framework of Euclidean geometry.

Exhaustionsbeweis zu (1):

Let $|A|$ be the area of the circle and $|B|$ the sum of all inscribed resp. circumscribed areas of triangles.

[3] Archimedis Opera omnia, Ed. Heiberg und Menge, Band 1, Leipzig 1880, pp. 258–271.

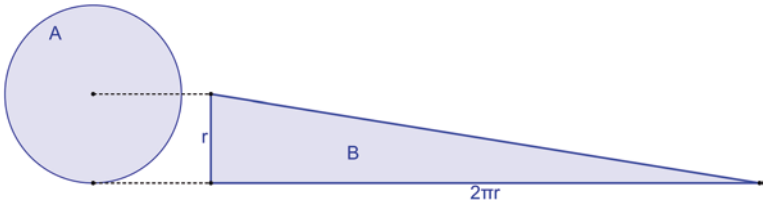


Fig. 12.7 Figure to Theorem 1 of Circle Measurement

a) Assumption $|A| > |B|$
Any number of isosceles triangles are inscribed on the circle in such a way that the apex lies at the center of the circle and the base a_i of each is a chord of the circle; the sum of all bases a_i forms an inscribed polygon P. According to the assumption, it holds for the area $|P| > |B|$. Of all the altitudes h_i of the triangles, there is one greatest altitude h_{max} among them $h_i \leq h_{max} \leq r$

$$|P| = \sum_1^n \frac{1}{2} a_i h_i \leq \sum_1^n \frac{1}{2} a_i h_{max} = \frac{1}{2} h_{max} \sum_1^n a_i < \frac{1}{2} r \cdot 2\pi r = |B|$$

This is a contradiction to the assumption.

b) Assumption $|A| < |B|$
An arbitrary number of triangles are circumscribed in the circle so that all the vertices of the triangles lie at the center of the circle and the base of each triangle is a section of a circle tangent; the sum of all the bases b_i forms an circumscribed polygon Q. According to the assumption, it holds for the area $|Q| < |B|$. Since the altitudes h_i of the inscribed triangles are equal to the radius r, it follows

$$|Q| = \sum_1^n \frac{1}{2} b_i h_i = \frac{1}{2} r \sum_1^n b_i > \frac{1}{2} r \cdot 2\pi r = |B|$$

This is also a contradiction to the assumption. Thus, $|A| = |B|$.

Archimedes was the first to use nested intervals to calculate π. For this, he used regular polygons, which he inscribed and circumscribed to the unit circle. Starting with the hexagon, he continued with the doubling of the vertices up to the 96-gon. If the length of the inscribed is set equal to the radius 1, then the circumscribed has the length of the sides $\frac{2}{3}\sqrt{3}$. In the first case, Archimedes received for the 96-gon $\frac{1}{2}U = \frac{96 \cdot 66}{2017\frac{1}{4}}$; in the second case $\frac{1}{2}U = \frac{96 \cdot 153}{4673\frac{1}{2}}$. By suitable rounding, he simplified the inequality to

$$3\frac{10}{71} < \pi < 3\frac{10}{70}$$

Since Archimedes does not know trigonometry, the calculation of the upper and lower limits is extremely laborious and extensive. The calculation can be found in the Latin edition of "De Mensura Circuli" by Gerard von Cremona[4] .

12.5 From the Book of Spirals

In his work *De lineis spiralibus*[5] Archimedes defines the spiral he found as the path of a point that performs both a rotational movement (with constant angular velocity ω) and a radial movement outward with constant velocity v (Fig. 12.8). Then it applies

$$v = \frac{r}{t} \;\therefore\; \omega = \frac{\varphi}{t}$$

Division with a constant $a = \frac{v}{\omega}$ yields the polar form

$$\frac{r}{\varphi} = a \Rightarrow r = a\varphi$$

A striking property of the Archimedean spiral is its constant *pitch:* If you draw a radius vector from the center, the intersections with the spiral have the constant distance $2\pi a$. This is the content of Theorem 12.

Theorem 12
Every line drawn from the center to the spiral intersects the turns at the same angle and the distances of the intersection points are equal.

Theorem 18
The tangent at point B cuts a section on the vertical axis whose length is equal to the circumference of the radius $r = |AB|$ (Fig. 12.9).

With modern means, one can verify the claim as follows. The slope in polar coordinates is

$$y' = \frac{r' \tan\varphi + r}{r' - r\tan\varphi}$$

With $r = a\varphi$ and $r' = \frac{dr}{d\varphi} = a$ it follows at point $B(2\pi a|0)$ with $\varphi = 2\pi$

$$y' = \frac{a\tan 2\pi + 2\pi a}{a - 2\pi a \tan 2\pi} = 2\pi$$

The tangent equation is thus

[4] Archimedes, Gerard von Cremona (Ed.): De Mensura Circuli, in the anthology Clagett pp. 41–58.

[5] Archimedis Opera omnis, Ed. Heiberg und Menge, Vol. 2, Leipzig 1881, pp. 1–139.

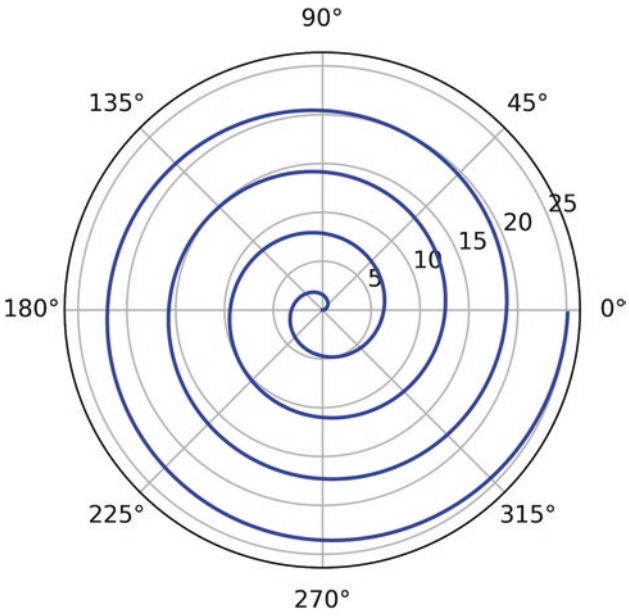
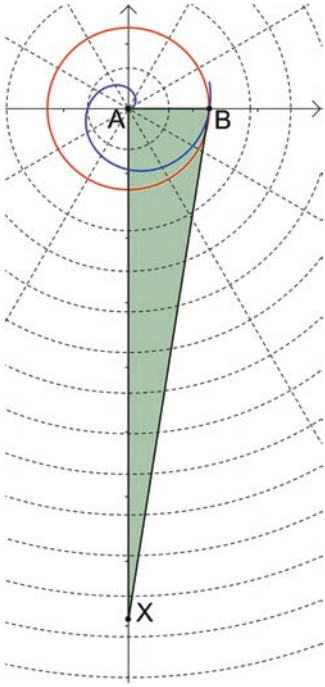


Fig. 12.8 The Archimedean Spiral

Fig. 12.9 Tangent to spiral



$$t{:}y = \underbrace{y(2\pi a)}_{=0} + y'(x - 2\pi) = 2\pi x - 4\pi^2$$

The tangent section results from this for $x = 0$ to: $|AX| = 4\pi^2$. This is equal to the circumference of the circle with radius $r = |AB|$

$$U = 2\pi r = 2\pi \cdot 2\pi = 4\pi^2$$

Theorem 19 shows that the tangent sections in the nth point of the horizontal axis are equal to the circumference of the nth circle.

In Theorem 25, Archimedes succeeds in measuring the area K, which is enclosed by the first winding. Since the derivation of Archimedes is comprehensive and dependent on many other lemmas, the modern method is chosen here.

Theorem 25

It is to be shown that the area described by the spiral in the first rotation is equal to one third of the area of the circle with radius $r = |AB|$ (Fig. 12.10).

Proof: The sector formula by G.W. Leibniz provides for the first winding area

$$K = \frac{1}{2}\int_0^{2\pi} r^2 d\varphi = \frac{1}{2}\int_0^{2\pi} (a\varphi)^2 d\varphi = \frac{1}{6}a^2[2\pi]^3 = \frac{4}{3}a^2\pi^3$$

This is equal to one third of the area of the circle A_1 with radius $r = |AB| = 2\pi a$

$$\frac{1}{3}A_1 = \frac{1}{3}\pi r^2 = \frac{4}{3}a^2\pi^3$$

In the course of the proof it is also shown that the sum of areas $K+L$ of the first two circulations is in the ratio 12:7 to the second circle area A_2.

$$K + L = \frac{1}{2}\int_{2\pi}^{4\pi} r^2 d\varphi = \frac{1}{6}a^2\left[(4\pi)^3 - (2\pi)^3\right] = \frac{28}{3}a^2\pi^3$$

The second circle area A_2 from the radius $r = |AC| = 4\pi a$ results in

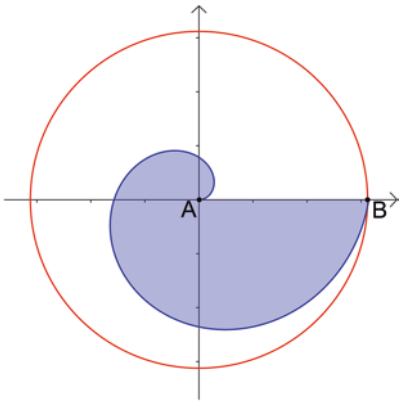


Fig. 12.10 Area of the first spiral rotation

$$A_2 = \pi(4\pi a)^2 = 16a^2\pi^3$$

The area ratio is

$$\frac{K+L}{A_2} = \frac{\frac{28}{3}a^2\pi^3}{16a^2\pi^3} = \frac{7}{12}$$

For the ratio of the first two winding areas follows

$$\frac{L}{K} = \frac{(K+L)-K}{K} = \frac{K+L}{K} - 1 = \frac{\frac{28}{3}a^2\pi^3}{\frac{4}{3}a^2\pi^3} - 1 = 7 - 1 = 6 \Rightarrow K = \frac{1}{6}L$$

The second winding area is six times larger than the first (Fig. 12.11).

Theorem 27
Here Archimedes provides a recursive relationship for the further winding areas

$$M = 2L \ \therefore \ N = 3L \ \therefore \ O = 4L \quad usw.$$

The ratios of these winding areas behave like whole numbers

$$L{:}M{:}N{:}O{:}\cdots = 1{:}2{:}3{:}4{:}\cdots$$

In addition to the area calculations, Archimedes also provides the tangent gradients.

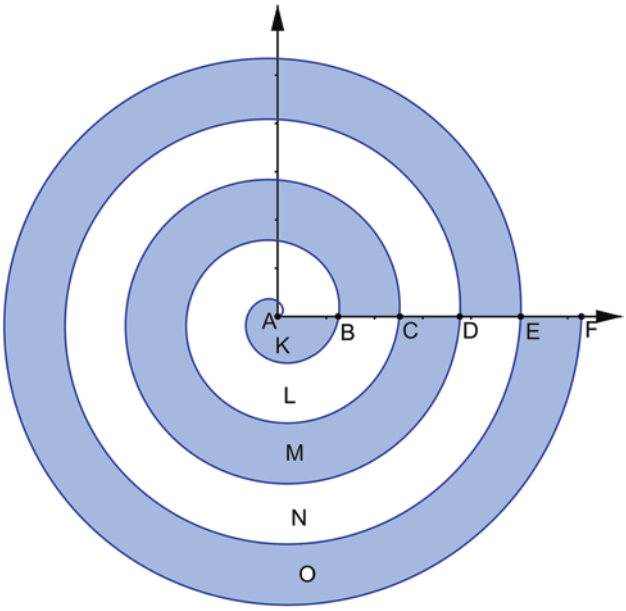


Fig. 12.11 The first 5 winding areas

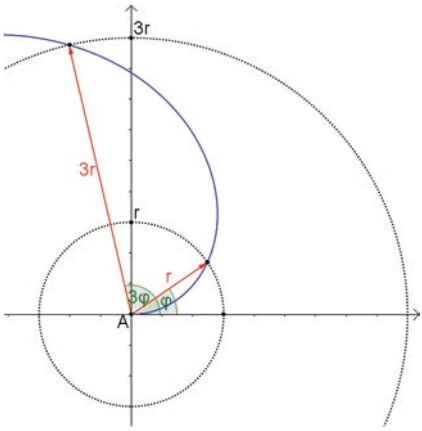


Fig. 12.12 Angle trisection by Pappos

Additions by Pappos

Pappos recognized that using the Archimedean spiral every angle can be divided into any number of parts (Fig. 12.12).

$$r\left(\frac{\varphi}{n}\right) = a\frac{\varphi}{n} = \frac{1}{n}r(\varphi)$$

Therefore, the radius vector of the angle must be divided accordingly. This is a consequence of *Collectio* IV, 20. In his *Collectio* IV, 22 Pappos also shows that the areas A, B, C, D within the first winding behave like 1:7:19:37 (Fig. 12.13).

Integration with polar coordinates in general terms yields

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\varphi = \frac{1}{6}a^2\left[\beta^3 - \alpha^3\right]$$

In the first quadrant follows

$$A = \frac{1}{6}a^2\left[\left(\frac{\pi}{2}\right)^3 - 0^3\right] = \frac{1}{48}a^2\pi^3$$

Analogously for the second

$$B = \frac{1}{6}a^2\left[\pi^3 - \left(\frac{\pi}{2}\right)^3\right] = \frac{7}{48}a^2\pi^3$$

In the third quadrant follows

$$C = \frac{1}{6}a^2\left[\left(\frac{3\pi}{2}\right)^3 - \pi^3\right] = \frac{19}{48}a^2\pi^3$$

Finally, in the fourth

$$D = \frac{1}{6}a^2\left[(2\pi)^3 - \left(\frac{3\pi}{2}\right)^3\right] = \frac{37}{48}a^2\pi^3$$

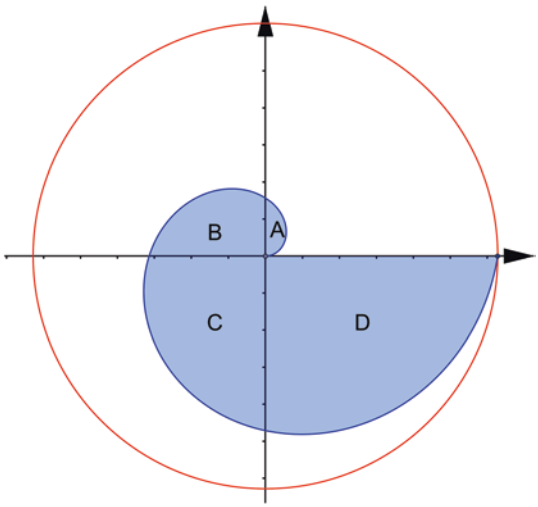


Fig. 12.13 Division into quadrants

Thus, the claim is proven. Pappos proves this with the proportions of the third powers of the radius vectors. Area A is related to the whole winding area as

$$\frac{A}{A+B+C+D} = \left[\frac{r\left(\varphi=\frac{\pi}{2}\right)}{r(\varphi=2\pi)}\right]^3 = \left[\frac{a\frac{\pi}{2}}{2\pi a}\right]^3 = \frac{1}{64}$$

For area B first results in ratio

$$\left[\frac{r(\varphi=\pi)}{r(\varphi=2\pi)}\right]^3 = \left[\frac{a\pi}{2\pi a}\right]^3 = \frac{1}{8} = \frac{8}{64}$$

Since the radius vector for B also intersects A, the share of A must be subtracted

$$\frac{B-A}{A+B+C+D} = \frac{7}{64}$$

For the other areas, the proof of Pappus' claim proceeds analogously.

12.6 The Book of Lemmata

The well-known figures of Salinon (σαλινον) or Arbelos (αρβηλος) and the division of angles by Archimedes can be found in the *Book of Lemmata*[6] (Latin *Liber Assumptorum*), which was translated into Arabic by Thābit ibn Qurra in the ninth century and later edited and commented on by Ahmad an-Nasawi. In 1661, the

[6] Archimedis Opera Omnia, Ed. Heiberg und Menge, Buch II, Teubner 1881, S. 428–446.

text was translated into Latin by A. Ecchellensis and published by G. A. Borelli as "Archimedes Liber Assumptorum" in his work *Apollonii Pergaei Conicorum*. The work is only preserved in Arabic. The work cannot directly originate from Archimedes, since the name Archimedes occurs several times and Archimedes never spoke of himself in the third person. T. Heath (1981, II, 75) is of the opinion that the originality of the figures *Arbelos, Salinon* or the division of angles speaks for the authorship of Archimedes.

Some of the important lemmas will be discussed here.

Lemma 1

If two circles touch each other in a point A and the two diameters BD and EF are parallel, then the points A, D, F are collinear (Fig. 12.14).

This is a simple stretching with the center A.

Lemma 4

Lemma 4 deals with the well-known figure of the *Arbelos* (= cobbler's knife). In the Thales circle of a right-angled triangle ABC, semi-circles are drawn over the two hypotenuse sections (radii R and r) and the circle over the altitude as diameter. It is to be shown that the circle is equal in area to the Arbelos; i.e. the area between the semi-circles (Fig. 12.15).

Modern proof: According to the altitude theorem, $h^2 = 2R \cdot 2r$; the circle area results in

$$\pi \left(\frac{h}{2}\right)^2 = \pi Rr$$

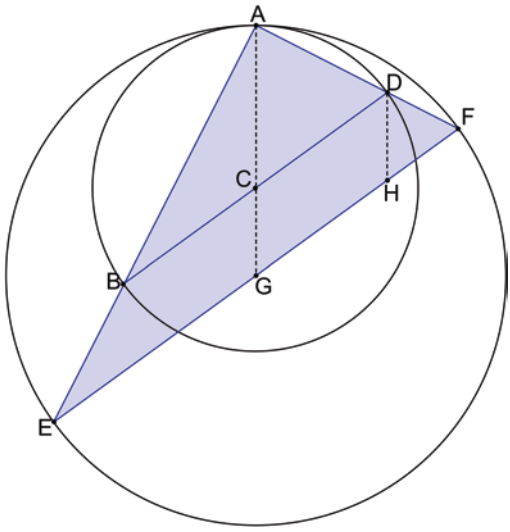


Fig. 12.14 To Lemma 1

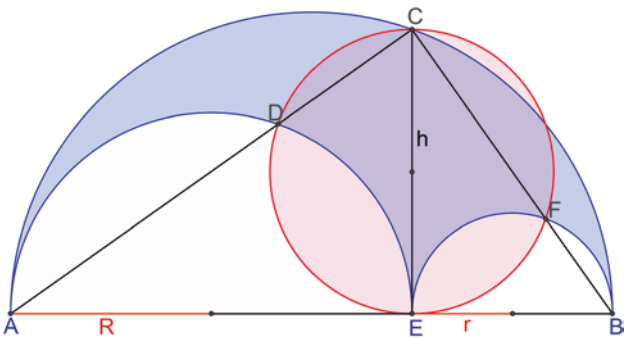


Fig. 12.15 To Lemma 4

The area of the Arbelos results as the difference of the semi-circle areas

$$\frac{1}{2}\pi(R+r)^2 - \frac{1}{2}\pi R^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left[(R+r)^2 - R^2 - r^2\right] = \frac{1}{2}\pi \cdot 2Rr = \pi Rr$$

The areas are therefore equal.

Lemma 5
If you inscribe two circles in the Arbelos figure that touch the height of the triangle, the circles are equal in area. The two circles are therefore also called *twin circles* of Archimedes (Fig. 12.16).

Proof: The left twin circle has the points of contact F, G and the diameter HE perpendicular to CD. According to Lemma 1, the points F, H, A or F, E, B therefore lie on a straight line. Because of the contact in G from the outside, it also follows from Lemma 1 that the points H, G, C or E, G, A are collinear in each case. Since F, I lie on the Thales circle over AB, the angles EFD and EID are right.

In the triangle ABD, E is the altitude point; thus, the line segment GI forms a common perpendicular of CH or BD. Since $CH \parallel BD$, $\triangle ACH$ is similar to $\triangle ABD$ and the proportion

$$\frac{|AD|}{|DH|} = \frac{|AB|}{|BC|}$$

applies.
Since $\triangle ACD$ is similar to $\triangle HED$, it follows analogously

$$\frac{|AC|}{|HE|} = \frac{|AD|}{|DH|}$$

In total, for the left twin circle

$$\frac{|AC|}{|HE|} = \frac{|AB|}{|CB|} \Rightarrow |AC| \cdot |CB| = |AB| \cdot \underbrace{|HE|}_{2x}$$

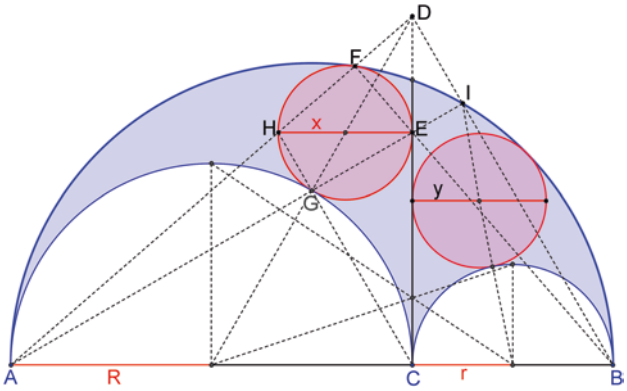


Fig. 12.16 To Lemma 5

Analogously, for the right circle

$$|AC| \cdot |CB| = |AB| \cdot 2y$$

It follows that the radii $x = y$; the two twin circles have the same area. The radii are given by

$$x = y = \frac{|AC| \cdot |CB|}{2|AB|} = \frac{2R \cdot 2r}{2 \cdot 2(R + r)} = \frac{Rr}{R + r}$$

This is half the harmonic mean of the two semicircle radii.

Lemma 6
The Arbelos figure is inscribed with a circle that touches the surrounding circles. The ratio of the diameter of the inscribed circle GH to the diameter of the large semicircle AB is sought (Fig. 12.17).

Proof: According to Lemma 1, the points A, G, D or B, H, D are collinear. Analogously, the points A, E, H or C, E, G lie on a line, as do B, F, G or C, F, H. The intersection of the line AD with the left semicircle (over AC) is I, corresponding to K the intersection of BD with the right semicircle (over BC). The intersection of IC with AE is L, corresponding to M the intersection of CK with BG. The intersections of the extensions of GL or HM with AB are N or P.

Since I and E lie on the Thales circle over AC, AE or IC represent heights in the triangle AGC. The line GN goes through the height intersection point L and is therefore itself a height; GN is therefore a perpendicular on AB. Also CK and BF are heights in the triangle CBH. The line HP goes through the height intersection point M; HP is therefore also a perpendicular on AB. Also, the angles at I, K, and D are right angles; DK is the common perpendicular of GD or CK. Therefore $CK \parallel GD$ and $CI \parallel DK$. Therefore, the triangles ACL and ABH are similar. The proportions therefore apply

$$\frac{|AC|}{|CB|} = \frac{|AL|}{|LH|}$$

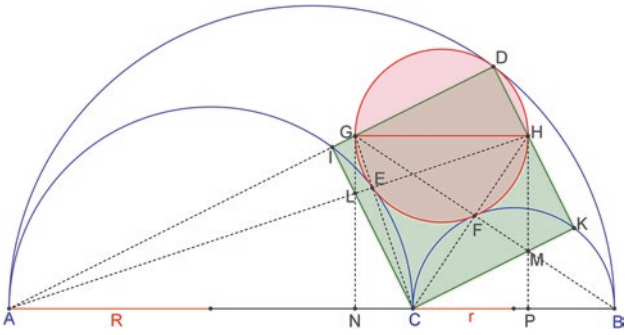


Fig. 12.17 For Lemma 6

Since the triangles ALN and AHP are also similar, it follows

$$\frac{|AL|}{|LH|} = \frac{|AN|}{|NP|}$$

It can be proven analogously

$$\frac{|BC|}{|CA|} = \frac{|BM|}{|MG|} = \frac{|BP|}{|PN|}$$

The overall relationship is

$$\frac{|AN|}{|NP|} = \frac{|NP|}{|BP|} \Rightarrow |NP|^2 = |AN| \cdot |PB|$$

$|NP|$ is therefore the geometric mean of the sections $|AN|$ and $|PB|$. The ratio $\frac{|AC|}{|CB|}$ of the given semi-circle diameter is x. From the above equations follows

$$|AN| = x|NP|$$

$$|NP|^2 = |AN| \cdot |PB| \Rightarrow |NP| = x|BP| \Rightarrow |BP| = \frac{1}{x}|NP|$$

This provides

$$|AB| = |AN| + |NP| + |PB| = |NP|\left(x + 1 + \frac{1}{x}\right)$$

For the desired ratio of the diameters, the following results

$$\frac{|GH|}{|AB|} = \frac{|NP|}{|AB|} = \frac{|NP|}{|NP|\left(x + 1 + \frac{1}{x}\right)} = \frac{x}{x^2 + x + 1}$$

Lemma 8
Lemma 8 includes the angle division by Archimedes, which is in section 11.3.1.

Lemma 9
If two secants AB and CD (not meeting the center) intersect at a right angle, then the sum of the opposite arcs is equal (Fig. 12.18):

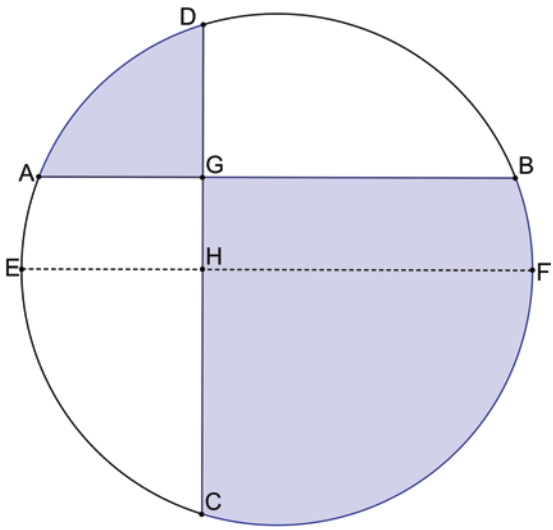


Fig. 12.18 To Lemma 9

$$\text{arc}(AD) + \text{arc}(BC) = \text{arc}(AC) + \text{arc}(BD)$$

Proof: The intersection of the secants is G. The diameter EF is drawn parallel to AB of the circle. The diameter EF intersects the second secant CD at a right angle in the point H, the center of CD. EF halves the circle; the semicircular arcs EDF and ECF are congruent. Because of $AB \parallel EF$

$$\text{arc}(ED) = \text{arc}(EA) + \text{arc}(AD) = \text{arc}(BF) + \text{arc}(AD)$$

It follows that

$$\begin{aligned}\text{arc of semicircle}(EDF) &= \text{arc}(ED) + \text{arc}(DF) \\ &= \text{arc}(BF) + \text{arc}(AD) + \text{arc}(DF)\end{aligned}$$

Because arc $(DF) =$ arc (CF) it follows that

$$\text{arc}(AD) + \text{arc}(BF) + \text{arc}(FC) = \text{arc of semicircle}$$

Therefore, the remaining arcs must also add up to a semicircular arc

$$\text{arc}(DB) + \text{arc}(AC) = \text{arc of semicircle}$$

Therefore, the opposite circular arcs are congruent.

Lemma 11
If two circular arcs AB and CD intersect at a right angle at a point F that is not the center of the circle, then (Fig. 12.19)

$$|AF|^2 + |BF|^2 + |CF|^2 + |DF|^2 = (\textit{diameter})^2$$

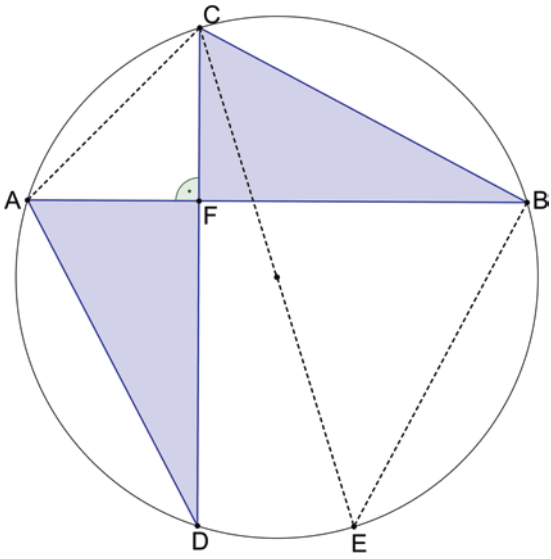


Fig. 12.19 To Lemma 11

Proof:
The diameter CE is drawn as an auxiliary line. The angle $\measuredangle CAB$ is congruent to $\measuredangle CEB$, since both are circumference angles to the arc CB. The triangles $\triangle CAF$ and $\triangle CEB$ are similar, since in addition to the angle at circumference, they also match in the right angles $\measuredangle AFC = \measuredangle CBE$. Thus, the remaining angles are also congruent

$$\angle ACF = \angle ECB$$

These angles are therefore circumference angles to the congruent chord $|AD| = |BE|$. It follows from Pythagoras

$$\begin{aligned}&\left(|AF|^2 + |DF|^2\right) + \left(|BF|^2 + |CF|^2\right) = |AD|^2 + |CB|^2\\ &= |BE|^2 + |CB|^2 = |AD|^2 + |CB|^2 = |CE|^2\end{aligned}$$

This is the claim.

Lemma 13
If the perpendiculars AF and BG are dropped from the endpoints of a circle diameter AB to a chord CD, then (Fig. 12.20):

$$|DF| = |GC|$$

Proof:
As a auxiliary line, the line AG is drawn and the parallel to AF through the center of the circle M. Intersections of the parallels with AG or GF are K or H. Since the triangles $\triangle AKM$ and $\triangle AGB$ are similar, K is also the center of AG. Since the triangles $\triangle GHK$ and $\triangle GFA$ are similar, H is also the center of FG. Since H is also the center of the chord CD, the claim $|CH| = |HD|$

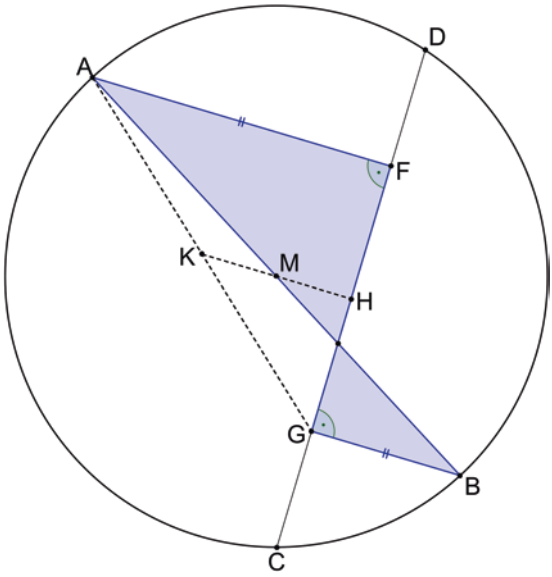


Fig. 12.20 To Lemma 13

Thus, the claim follows

$$|DF| = |DH| - |FH| = |CH| - |HG| = |CG|$$

Lemma 14

The following figure is called the *Salinon* (= salt cellar) of Archimedes: From the endpoints A, B of a semicircular diameter, two congruent segments $|AD| = |EB| = 2r$ are marked and two symmetrical semicircles are erected upwards. A semicircle is drawn downwards over the remaining segment $|DE| = 2R$. The arc midpoints of the semicircles over AB or DE are C or F. It is to be shown:

The area of the circle with the diameter CF is equal in area to the Salinon (Fig. 12.21).

Modern proof:

The area of the Salinon consists of the large semicircle with radius $R + 2r$ and the small semicircle with radius R, reduced by the small circle with radius r.

$$\begin{aligned}\frac{\pi}{2}(R+2r)^2 - \pi r^2 + \frac{\pi}{2}R^2 &= \frac{\pi}{2}\left[R^2 + 4Rr + 4r^2 - 2r^2 + R^2\right] \\ &= \pi\left(R^2 + 2Rr + r^2\right) = \pi(R+r)^2\end{aligned}$$

This is also the area of the circle over CF with radius $(R + r)$; this circle is also the circumscribed circle of the square $CIFH$, which is formed by the intersection points with the Salinon.

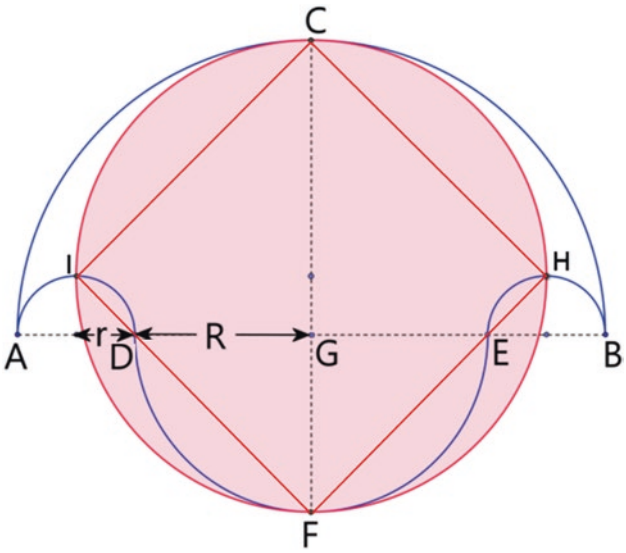


Fig. 12.21 To Lemma 14

12.7 The Quadrature of the Parabola

In the book *Quadraturae Parabolae*[7] Archimedes first determines the content of an area bounded by a parabola and thus anticipates a method of analysis. First, he proves two lemmas.

Theorem 19: Given is a parabolic segment ABC; through the center D of AC the parallel BD to the parabolic axis is drawn. If E is the center of AD, then the parallels $ZE \parallel BD$ and $ZF \parallel AD$ are drawn. Then it holds (Fig. 12.22):

$$|BD| = \frac{4}{3}|EZ|$$

Proof: According to the definition of the parabola it holds

$$\frac{|BD|}{|BF|} = \frac{|AD|^2}{|ZF|^2} = \frac{|AD|^2}{|ED|^2} = \frac{4|ED|^2}{|ED|^2} = 4 \Rightarrow |BF| = \frac{1}{4}|BD|$$

Because of $|BF| = |BD| - |FD|$ it follows

$$|ZE| = |FD| = |BD| - |BF| = |BD| - \frac{1}{4}|BD| = \frac{3}{4}|BD| \Rightarrow |BD| = \frac{4}{3}|ZE|$$

[7] Archimedis Opera omnia, Ed. Heiberg, Menge, Band 2, Leipzig 1881, pp. 291–353.

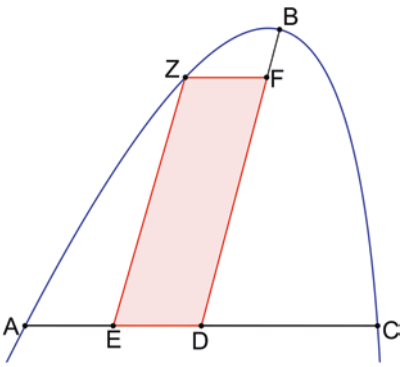


Fig. 12.22 Parabola, Theorem 19

Theorem 20: If a triangle with the same base and height is inscribed in a parabolic segment, then the triangle is larger than half of the segment.

At the same time it is shown that the triangle area is a lower bound, the parallelogram area is an upper bound for the area of the parabolic segment.

Theorem 21: If a triangle is inscribed in a parabolic segment and if similar triangles are inscribed in the remaining segments, then the triangle of the whole segment has eight times the area of a triangle of the remaining segment.

Let ABC be the parabolic segment; halve AC at D and draw BD parallel to the parabola axis. B is then the vertex of the segment; $\triangle ABC$ agrees in base line and height with the segment. Halve AD at E and draw EZ parallel BD. EZ cuts AB at the point F. Z is then the vertex of the remaining segment ABZ; $\triangle ABZ$ agrees in base line and height with the segment ABZ. It is to be shown that the area of $\triangle ABC$ is eight times that of $\triangle ABZ$ (Fig. 12.23).

Proof: By Theorem 19, it holds that: $|BD| = \frac{4}{3}|EZ|$. Because $EF \parallel BD$, it follows that

$$|BD| = 2|EF|$$

Thus follows

$$|EF| = \frac{2}{3}|EZ| \Rightarrow |EF| = 2|FZ|$$

Therefore, $\triangle AEF$ is twice as large as $\triangle AFZ$ and $\triangle ZFB$. This shows: $\triangle ABC$ is eight times as large as $\triangle AZB$. Analogously, $\triangle ABC$ is eight times as large as $\triangle BHC$. Therefore,

$$\triangle\, ABC = 4(\triangle\, AZB + \triangle\, BHC)$$

Theorem 22: If $A_1, A_2, A_3, \cdots A_n$ is a series of areas, each of which is four times as large as the following one and the largest area A_1 is the area of the triangle ABC, then

$$A_1 + A_2 + A_3 + \cdots + A_n < \mathcal{F}(segment\ ABC)$$

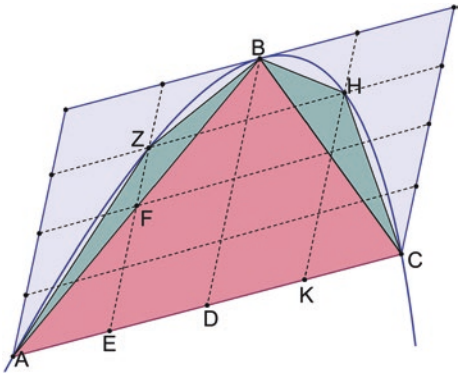


Fig. 12.23 Parabola, Theorem 21

Proof: The sum $A_1 + A_2 + A_3 + \cdots + A_n$ is the area of a polygon inscribed in the segment and is therefore smaller than the segment.

Theorem 23: If $A_1, A_2, A_3, \cdots A_n$ is a sequence of areas, each of which is an area A_i four times as large as the following one and the largest area A_1 is, then

$$A_1 + A_2 + A_3 + \cdots + A_n + \frac{1}{3}A_n = \frac{4}{3}A_1 \quad (*)$$

Proof: According to the finite geometric series known to Archimedes

$$A_1 + \frac{1}{4}A_1 + \left(\frac{1}{4}\right)^2 A_1 + \left(\frac{1}{4}\right)^3 A_1 + \cdots + \left(\frac{1}{4}\right)^{n-1} A_1 = \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}} A_1 = \left[\frac{4}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^{n-1}\right] A_1$$

This provides for the above sum (*)

$$\left[\frac{4}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^{n-1}\right] A_1 + \frac{1}{3}\left(\frac{1}{4}\right)^{n-1} A_1 = \frac{4}{3}A_1$$

Theorem 24: The area of the segment K is equal $\frac{4}{3}A_1$, where A_1 is the area of the inscribed triangle.

Proof: Assumption $K \neq \frac{4}{3}A_1$

Case 1: $K > \frac{4}{3}A_1$. Since the segment is greater than $\frac{4}{3}A_1$„ one can inscribe additional triangles in the remaining segments until the area of the inscribed polygon satisfies

$$A_1 + A_2 + A_3 + \cdots + A_m + \frac{1}{3}A_m = \frac{4}{3}A_1 > K$$

According to Theorem 22, this is a contradiction to the assumption.

Case 2: $K < \frac{4}{3}A_1$. If the segment is smaller than $\frac{4}{3}A_1$„ one can remove triangles from the remaining segments until the area of the inscribed polygon satisfies

$$A_1 + A_2 + A_3 + \cdots + A_k + \frac{1}{3}A_k = \frac{4}{3}A_1 < K$$

According to Theorem 22, this is a contradiction to the assumption. In total, it is shown that the area of the parabolic segment is $\frac{4}{3}$ times the triangle area that coincides with the segment in base line and height.

12.8 The Palimpsest

The Palimpsest, known as Codex C in the expert group, which was auctioned off at *Christie's* in New York on October 29, 1998, has a turbulent history. The book belonged – alongside two others – to a collection of parchment sheets with works by Archimedes, which were copied in Constantinople around 850 for the university founded in 863. The university was led by Leon the Geometer, who also had an Euclid and Ptolemy manuscript created. During the plundering of Constantinople caused by the 4th Crusade, initiated by Pope Innocent III, the Codex C, as well as the Codices A and B, were stolen in 1204.

The texts of the codices overlap: "On the Balance of Flat Surfaces", "Quadrature of the Parabola" can be found in A and B, "Sphere and Cylinder", "Measurement of the Circle" and "On Spiral Lines" are contained in A and C, "Floating Bodies" in B and C. "On Conoids and Spheroids" and "The Sand-Reckoner" are only available in A, "The Method" and "Stomachion" only in C. In Codex A, the scribe added a dedication:

Dedicated to Leo the Geometer, may you be successful and live many years, you friend of the Muses.

Codices A and B did not reappear until 1269 in the Vatican, where the Franciscan monk Wilhelm von Moerbeke translated some parts into Latin. Codex B disappears sometime after 1311. In 1496, Lorenzo de Medici wanted to purchase Codex A for his library, and the search led to Venice and the library of the humanist Giorgio Valla, who only allowed the creation of a copy, which today is located in the Bibliotheca Laurenziana in Florence. In 1531, the Valla library came into the possession of the Alberto Pio family. After the death of the last member, Ridolfo, in 1564, Codex A also disappeared.

In the meantime, Codex C ended up in the monastery *Mar Saba* (Palestine), where the parchment pages were scraped off and rewritten with prayers along with parchments from other manuscripts. The prayer book (Greek *Euchologion*) was completed on Easter Sunday, April 14, 1229, by a monk named Ioannes Myronas, whose dedication could be deciphered in the manuscript using UV light. There was a special occasion to celebrate Easter that year, as the army of Emperor Frederick II had recaptured Jerusalem the month before.

The Euchologion remained in the Greek Orthodox monastery *Mar Saba (To the Holy Sabba)* near Bethlehem until 1846, where the theologian K. von Tischendorf found it; he followed a report by the English clergyman George Croly from 1839. By the way, on this journey Tischendorf also discovered the *Codex Sinaiticus,* in the St. Catharine monastery on Mount Sinai, which is the oldest (known) manuscript of the New Testament.

The prayer book was newly catalogued in 1899 by the librarian *Papadopoulos-Kerameos*, as were all manuscripts that belonged to the Greek Orthodox Church in Jerusalem but were kept in the Metochion of the *Church of the Holy Sepulchre* in Constantinople. Kerameos recognised some lines of the underlying Archimedes text in the prayer book. This came to the attention of the Danish researcher J.L. Heiberg who examined the manuscript in the Metochion in 1906. He was able to photograph and study it extensively in 1908. In the work – now called Codex C – he could identify a total of seven different treatises. The work *On the Method* was completely new and he was able to decipher it with great difficulty. The publication of the work by Heiberg was a scientific sensation of the year 1907.

In 1938, the books of the Metochion were brought to the National Library of Athens, where Codex C disappeared during transport. The further fate of the prayer book remained unknown until it was delivered to Christie's for auction in October 1998 by the Frenchwoman A. Guersan. Probably her grandfather M.L. Sirieux, who was stationed in Greece during the First World War, had acquired it. The palimpsest had suffered extremely from moisture and mold in the last 90 years; In addition, the previous owners had tried to increase the value of the manuscript by four amateurishly painted images of saints. The Euchologion was auctioned off by an anonymous American manuscript collector for 2 million US$ and handed over to the *Walters Art Museum* in Baltimore for scientific evaluation. Under the leadership of the Israeli-American R. Netz, the project was sensationally marketed in all media (www.archimedespalimpsest.net).

Despite the now available modern technologies, it will be difficult to completely restore the erased text of the original labeling from the liturgical text of the thirteenth century. In addition to the alleged puzzle game *Stomachion* there is now an improved version of the *method;* Furthermore, new sketches were found for the treatises *On the Spiral* and *On Floating Bodies.*

12.9 The Stomachion

As an alleged novelty—so praised by Netz—the palimpsest offers the discovery of a puzzle, which, like the other mathematical content, probably originates from Archimedes. This so-called *Stomachion* (στόμαχος = stomach) consists of 14 puzzle pieces, which was already described in the work *Cento Nuptalis* (Book XVII) by the poet M. Ausonius (310–395 AD) under the name *ostomachion*; in Roman literature it is also called *Loculus* (Fig. 12.24). The Stomachion was already known before; it is already depicted in the Archimedes book by Dijsterhuis, which was compiled from articles in the journal *Archimedes* up to 1944. The Stomachion consists of 14 parts, which can be assembled in different ways to form a square. Two of the 14 pieces are twofold; these are numbers 6 and 7. Six puzzle pieces are designed so that they can only occur in combination; these are the pairs (1;2), (9;10) and (11;12).

It remains unclear what motivated Archimedes to treat a relatively simple puzzle. But the statement missing from Heiberg's edited text, which, together with

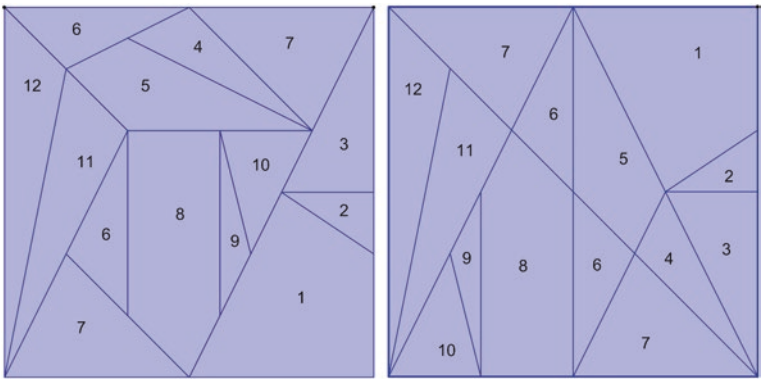


Fig. 12.24 Two dissections of the Stomachion

its proof, would justify its composition, can be found in an Arabic version of the Stomachion published by Heinrich Suter in 1899. There it says after the construction of the 14 parts of the square: *We now prove that each of the fourteen parts is in rational proportion to the whole square.* This statement fits well with Archimedes' effort to prove the proportions of different bodies such as spheres and cylinders, or of surfaces, to be rational.

Apparently, Netz saw the version contained in the Arabic version of the Stomachion as not spectacular enough for the expectations that had by now been aroused. Against this background, he considered the solution contained in the Arabic text to be too trivial for a genius like Archimedes. So he came up with the idea of presenting the Stomachion as a combinatorial problem: In how many ways can the puzzle pieces be assembled to form a square? R. *Netz*[8] commissioned the computer specialist Bill Cutler (Illionis) to solve the riddle. With the help of his program, he was able to show that there are exactly 17,152 ways to assemble the puzzle pieces into a square. Without rotations, there are 536 basic solutions, with 32 rotations possible for each.

So is Archimedes also the inventor of combinatorics? There is no evidence whatsoever that Archimedes had combinatorial counting in mind in his writings. Netz refers to a sentence that is missing in Heiberg's translation. This passage is called: *There are therefore no small numbers of figures from the same.* To derive the justification that the Stomachion is a combinatorics problem does not seem justified.

Note: Since the controversy raised by S. Unguru, the mathematical-historical writings of the traditional authors are considered outdated. Netz, who studied with Unguru in Tel Aviv, tries to claim the right of interpretation over the future interpretation of Archimedes through the popularity he has gained through his

[8] Netz R., Noel W.: The Archimedes Codex, Weidenfeld & Nicolson 2007, p. 253.

public decipherment work on the Archimedes Codex. In his book *The Shaping of Deduction in Greek Mathematics* from 1999 he claims that almost all interpretations of Greek mathematics published before about 1975 are to be rejected either as wild speculation or as *unhistorical* works of art. He himself is planning a completely new edition of all Archimedean works. Two volumes of the edition have already been published by Cambridge[9] ,[10] .

In the aforementioned book from 1999, his explanation received great attention that the formal proofs of the Greeks had arisen through the labeling of the geometric figures with letters and their formal adoption in the text. He also introduced new statistical methods. For example, he found (p. 107) that the vocabulary used by Archimedes, despite the variety of topics, comprises only 851 different words. Furthermore, it turned out that 143 words from the most frequently used words in Archimedes' works can also be found in Euclid's *Data*. This could mean that Archimedes conceived his writings as a mathematical model text, similar to what Euclid (or his editor) tried in the *Elements* or in the book *Data*.

12.10 The Method, Theorem 2

Archimedes considers the configuration shown in mechanical equilibrium with respect to the point of rotation A. The inscribed circle (center O) has the perpendicular diameters $|AB|$ and $|CD|$„ the line AC cuts the line through B at point E, the line AD cuts at point F. $ABEH$ is a square, $EFGH$ is a rectangle. If r is the radius of the inscribed circle, then:

$$|EB| = |BF| = 2r$$

The angles $\measuredangle EAB$ and $\measuredangle FAB$ each amount to 45°. The rectangle is cut by the line PQ perpendicular to AB; the point of intersection with the axis AB is X. The other points of intersection are R, S with the circle, T with AE. As in Fig. 12.25 we set

$$|XT| = x \ \therefore \ |XR| = y$$

Since $\triangle AXT$ is isosceles-right angled, it follows that

$$|AX| = x \Rightarrow x^2 + y^2 = |AR|^2 \tag{12.1}$$

The $\triangle ARB$ located in the Thales circle is right angled; according to the Pythagorean theorem, it holds that

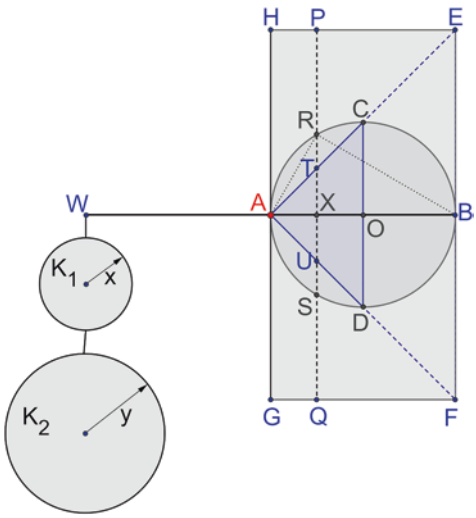
$$|AR|^2 = x \cdot 2r \tag{12.2}$$

The length of the left lever arm $|WA| = 2r$. This explains the construction.

[9] Netz R.: The works of Archimedes, Volume 1: The Two Books On the Sphere and the Cylinder, Cambridge University Press 2004.

[10] Netz R.: The works of Archimedes, Volume 2: On Spirals, Cambridge University Press 2017.

Fig. 12.25 Method, set 2



The entire configuration is now rotated around the axis WB: The inscribed circle generates a sphere with a radius of r„ the triangle EAF generates a cone with the base circle radius $|BE| = 2r$ and the height $|AB| = 2r$.. The rectangle $EFGH$ generates a cylinder with the base circle diameter $|BE| = 2r$ and the height $|AB| = 2r$.. With this rotation, the line PQ generates a plane that intersects the cylinder in a circle with a radius of $2r$, the cone in a circle K_1 with a radius of x, and the sphere in a circle K_2 with a radius of y.. Dividing Gl. (12.1) by $4r^2$ results in Gl. (12.1) and (12.2)

$$\frac{x^2 + y^2}{4r^2} = \frac{|AR|^2}{4r^2} = \frac{x}{2r}$$

Expanding the left side yields the equation

$$\frac{\pi\left(x^2 + y^2\right)}{\pi\left(2r\right)^2} = \frac{x}{2r} \tag{12.3}$$

If one considers the line WB as a lever around the point of rotation A, then Eq. (12.3) can be interpreted as follows: The ratio of the surface areas of K_1 and K_2 (cut out of the cone and the sphere) to the area of the circle with radius $2r$ (cut out of the cylinder) is equal to the ratio of the distance $|AX| = x$ to the distance $|AW| = 2r$.. If one imagines PQ running from HG to EF, then the resulting slices can be assembled into a sphere and a cone. With the center of gravity of the cylinder O, the lever balance yields the ratio

$$\frac{sphere + cone}{cylinder} = \frac{|AO|}{|AW|} = \frac{1}{2} \Rightarrow 2(sphere + cone) = cylinder$$

Since, according to [Euclid XII, 12], the volume of the cylinder is three times that of the inscribed cone, it follows that

$$2 \cdot sphere = cone\ EAF$$

The cone ADC has half the height and half the base radius of cone EAF, so it follows that

$$cone\ EAF = 8 \cdot cone\ CAD \Rightarrow sphere = 4 \cdot cone\ CAD$$

Result: The volume of the sphere is thus four times that of the cone that coincides with the sphere in base radius and height.

Conclusion: If the height of the cone is equal to the diameter of the sphere, then its volume is equal to half the volume of the sphere. Thus, the volume of the sphere is the $\frac{2}{3}$-fold of the circumscribed cylinder.

12.11 Archimedes' Tomb Figure

Inscribing a sphere in a cylinder (of the height of the sphere's diameter) yields a geometric body that Archimedes particularly valued and therefore immortalized on his tomb. The tomb that Archimedes received after his violent death during the conquest of Syracuse had fallen into oblivion. The siege of Syracuse (Sicily) in 212 BC by the Romans during the Second Punic War is described in detail by Plutarch. Although the commander *Marcellus* had given the explicit order to spare the scholars, Archimedes was killed by a Roman soldier when he was busy with an abacus, according to tradition. The execution is depicted in the mosaic of Fig. 12.26, which is a copy of the famous mosaic in Naples. Plutarch (Vitae Parallelae, Life of Marcellus, 18) reports another version: *As Archimedes was carrying mathematical instruments, dials, spheres, and angles, by which the magnitude of the sun might be measured to the sight, some soldiers seeing him, thinking that he carried gold in a vessel, slew him.*

This mosaic has a curious history: it probably came from the possession of Jérôme Bonaparte, the youngest brother of Napoléon Bonaparte, was auctioned off after his death (1860) and is now part of the inventory of the Städel Museum in Frankfurt am Main. Since the origin of the mosaic was initially unknown and it was heavily damaged, it was thought to be from antiquity. However, closer examination revealed that it is a newer artifact from the eighteenth century.

The reason for the soldier's crime could be that he wanted to bring the devices made of bronze by Archimedes, an astronomical clock (probably a planetarium) and a celestial globe, into his possession. The celestial globe came as war booty to Rome to the Temple of Vesta, where Ovid admired it in 8 BC and wrote a poem about it. The clock remained in the family of Marcellus, so that the astronomer Gaius S. Gallus, who saw it, could predict the solar eclipse of June 21, 168 BC (the visibility in Rome was not predictable). Cicero reports in *De re publica* I, 14 about a performance of the devices by Gallus in the house of Marcellus' grandson:

> While he was staying in the house of Marcus Marcellus, who had been his colleague in the consulship, asked to see a celestial globe which Marcellus's grandfather had saved

Fig. 12.26 Mosaic of the death of Archimedes (Wikimedia Commons)

> after the capture of Syracuse ...which I had often heard mentioned on account of the great fame of Archimedes; but its appearance, however, did not seem to me particularly striking... But as soon as Gallus had begun to explain, in a most scientific manner, the principle of this machine, I felt that the Sicilian geometrician must have possessed a genius superior to anything we usually conceive to belong toour nature. For Gallus assured us that that other solid and compact globe was avery ancient invention, and that the first model had been originally made by Thales of Miletus. That afterward Eudoxus of Cnidus, a disciple of Plato, had traced on its surface the stars that appear in the sky... He added that the figure of the globe, which displayed the motions of the sun and moon, and the five planets, or wandering stars, could not be represented by the primitive solid globe; and that in this the invention of Archimedes was admirable, because he had calculated how a singlerevolution should maintain unequal and diversified progressions in dissimilarmotions. In fact, when Gallus moved this globe, we observed that the moon succeeded the sun by as many turns of the wheel in the machine as days in the heavens. From whence it resulted that the progress of the sun was marked as in the heavens, and that the moon touched the point where she is obscured by the earth's shadow at the instant the sun appears opposite.

At that time, there were only very few astronomical clocks, one of which was built by *Posidonius* on Rhodes, where Cicero (*De natura Deorum* II, 34) could admire it on his journey to Asia Minor (79-77 BC). It is possible that the famous mechanism found in 1900 in the sea off the island of Antikythera was the clock of Posidonius and was lost in a shipwreck. The coins found on board come from Pergamon from the period 86-67 BC. This suggests that the ship left Pergamon during this period and probably made a stopover on Rhodes. The ship probably sank between 70 and 60 BC. The cargo of Greek statues found on board was obviously intended for Rome. A new analysis of the mechanism by J. Evans of the

University of Puget Sound suggests that it was created around 205 BC. A. Jones of New York University has made further progress in deciphering the inscriptions; they were able to identify a total of 3500 characters, some of them in Corinthian script.

The mechanism of Antikythera consists of 30 bronze gears driven by springs, which are said to have realized the calendar cycle of the Athenian astronomer *Meton* (432 BC), a cycle of 19 years and 235 lunar orbits. The cycle of Kallippos with 76 years is also said to have been realized; as well as an indication of the Olympiad.

The two astronomical clocks of Archimedes as well as the planetarium of Antikythera were made of a metal like bronze. The production of such an instrument required, in addition to mathematical-astronomical knowledge, fine-mechanical and above all metallurgical knowledge. For this purpose, there must have been the necessary metalworkers and workshops in Syracuse; this throws a completely new light on the craft skills of the Greeks around 220 BC.

It was Cicero who, as quaestor of Sicily in 75 BC, had Archimedes' tomb searched for in Syracuse and recognized it by the aforementioned figure (Tusc. Disp., V, 23).

> I remembered some verses, which I had been informed were engraved on his monument, and these set forth that on the top of the tomb there was placed a sphere with a cylinder. When I had carefully examined all the monuments (for there are a great many tombs at the Agrigentine gate), I observed a small column standing out a little above the briers, with the figure of a sphere and a cylinder upon it; whereupon I immediately said to the Syracusans ... that I imagined that was what I was inquiring for… When we could get at it, and had come near to the front of the pedestal, I found the inscription, though the latter parts of all the verses were effaced almost half away. Thus one of the noblest cities of Greece, and one which at one time likewise had been very celebrated for learning, had known nothing of the monument of its greatest genius, if it had not been discovered to them by a native of Arpinum [=Cicero].

The tombstone is said to have been rediscovered in 1995 during the construction of a hotel in Syracuse.

Why did Archimedes choose the cylinder with inscribed sphere as a tomb figure?

In his work *On the Method,* which was only rediscovered by Heiberg in 1906 and dedicated to Eratosthenes, Archimedes had formulated Theorem 2 as follows:

> The cylinder, whose base is equal to the great circle of the sphere and whose height is equal to the diameter of the sphere, is $\frac{3}{2}$ of the sphere in area.

This allowed Archimedes to determine the surface area of the cylinder O_{cyl}. That the surfaces are in the ratio of 3:2 is easy to see

$$\frac{O_{cyl}}{O_{sph}} = \frac{4\pi R^2 + \pi R^2 + \pi R^2}{4\pi R^2} = \frac{3}{2}$$

Remarkably, this is also the ratio of the volumes

$$\frac{V_{cyl}}{V_{sph}} = \frac{2\pi R^3}{\frac{4}{3}\pi R^3} = \frac{3}{2}$$

The surface area of the cylinder even agrees with the surface area of the sphere!

$$\frac{M_{cyl}}{V_{sph}} = \frac{4\pi R^2}{4\pi R^2} = 1$$

If one adds the inscribed cone, one obtains the beautiful proportion for the volumes, which would surely have pleased the heart of Pythagoras

$$V_{con}:V_{sph}:V_{cyl} = \frac{2}{3}\pi R^3:\frac{4}{3}\pi R^3:2\pi R^3 = 1:2:3$$

12.12 More Works by Archimedes

In addition to his mathematical treatises, some physical findings should also be mentioned.

a) Buoyancy
Archimedes was supposedly commissioned by King Hieron of Syracuse to non-destructively test the gold content of a crown. Archimedes is said to have discovered the principle of buoyancy while getting into a bath, as Vitruvius writes at the beginning of Book IX of his *Architecture.* He supposedly jumped out of the bath and ran naked into the street, yelling *Eureka!* By measuring the displaced water when the crown or a lump of gold of equal weight was submerged, Archimedes discovered that the crown was not made of pure gold. The precise formulation of Archimedes' principle, which appears in the book *On Floating Bodies,* is: *The buoyant force on a submerged object is equal to the weight of the displaced fluid.*

b) Lever Law
The Lever Law is formulated by Archimedes in the book *On Plane Bodies*; it finds numerous applications in the proof experiments using mechanics. With this he determined the center of gravity of rectangles, trapezoids and segments of the circle or the parabola. Pappos transmitted in Book VIII, 11 the saying that Archimedes is said to have made when he discovered the lever effect:

> Give me a fixed point and I will lift the earth out of its hinges!

c) Pulley
The inventor of the pulley is unknown; however, the invention is attributed to Archimedes. Plutarch reports in his Marcellus report (from *Vitae parallelae*) of a ship that King Hieron of Syracuse had built for Ptolemy I. All dock workers could

not pull the ship out of the dock. Hieron was able to pull the entire ship into the water by himself using a pulley mounted by Archimedes. The king is said to have said: *From this day on, one should believe the words of Archimedes in everything.*

d) Water Screw

The invention of the water screw took place during his stay in Egypt, as Diodorus Siculus *(Bibl. Hist. V.37.3)* writes.

The standard work of Archimedes was edited by Heiberg[11] Ivo Schneider[12] brings additional material to Archimedes' work in optics and astronomy. The Archimedes book by Günter Aumann[13] is a textbook that goes into many mathematical details that cannot be presented here.

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Eratosthenes of Cyrene

13

Eratosthenes (Fig. 13.1) (Ἐρατοσθένης ὁ Κυρηναῖος) from Kyrene (today Shahat/Libya) lived from 273-192 BC and was a contemporary, letter-writer, and contact for the Alexandrian mathematician Archimedes. Archimedes dedicated his writing *The Method:*

> Since I see that you are a competent scholar and not only an outstanding teacher of philosophy, but also an admirer [of mathematical research].

Since Strabon (*Geographica,* 15) connects him with some students of the Peripatos (including Ariston of Chios and Zenon of Kition), it could assumed that he was educated in Athens. If we assumes that Zenon, the famous founder of the Stoics, died in 262, then the birth year must set earlier. But even the date of Zenon's death is disputed; according to Diogenes Laertios (VII, 6-28), it could have been as late as 256.

However, the Suda reports that the famous poet Callimachus was one of his teachers, who worked as an assistant to the librarian Zenodotus of Ephesus and later catalogued a large part of the manuscripts of the Alexandrian Library in this role. In 235 BC he was himself appointed chief librarian by Ptolemy III Evergetes (reign 247-222). Around 230 he was also entrusted with the education of the son and successor Ptolemy IV Philopator, who ascended the throne in 222.

From Strabo's writings we know that Eratosthenes spoke highly of his teachers in Athens. Whether he was actually in Athens is not certain. A comprehensive discussion can be found in K. Geus[1], whose habilitation thesis is a standard work on Eratosthenes (in German).

[1] Geus K.: Eratosthenes von Kyrene – Studien zur hellenistischen Kultur- und Wissenschaftsgeschichte, Münchener Beiträge zur Payprusforschung und antiken Rechtsgeschichte, Band 92, C. H. Beck, 2002.

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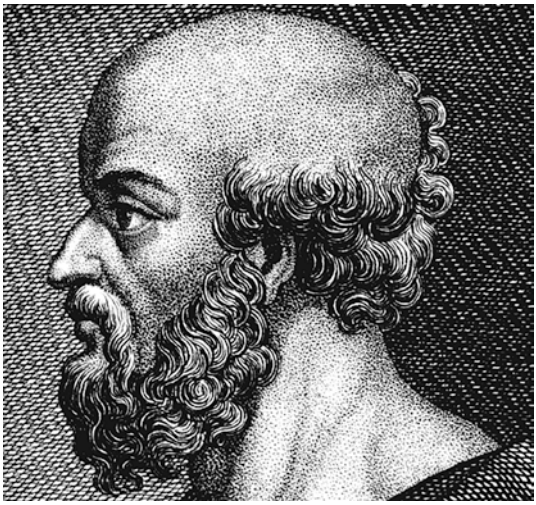

Fig. 13.1 Eratosthenes (Wikimedia Commons)

Eratosthenes was a true multi-talented man, writing works on philosophy, mathematics, geography, astronomy, history, and poetry. Unfortunately, none of his works have survived; however, we know the titles of some of them: *On Plato, On Comedy, On the Measurement of the Earth*, and *On the Constellations*. Other topics included the calendar, the chronology of Greece (since the Trojan War), and a chronicle of the Olympic Games. Some of his dates are widely accepted: the start of the Dorian migration in 1104/03 BC and First Olympiad in 777/76; his dating of the Trojan War is disputed: the fall of Troy in 1184/83 BC; Eratosthenes here reckons back 860 years from Alexander's death. Dionysos of Cyzicus wrote an epitaph for Eratosthenes in the Greek Anthology (VII,78): *A mild old age, no darkening disease, put out thy light, Eratosthenes son of Aglaus, and, thy high studies over, thou sleepest the appointed sleep. Cyrene thy mother did not receive thee into the tombs of thy fathers, but thou art buried on this fringe of Proteus' shore [=Alexandria], beloved even in a strange land.*

13.1 Eratosthenes as a Mathematician

The invention of the mesolabion (μεσολάβιον) attributed to Eratosthenes is a mechanical device used to determine the two middle proportions x and y (defined by $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$) of two given lengths a and b by sliding. The mesolabion also allows the problem of doubling a cube to be solved mechanically.

There is even an epigram dedicated by Eratosthenes to his king Ptolemy III on a pillar in front of the Serapeion, which solves the problem of doubling a cube. It is preserved along with two other of his epigrams in the *Greek Anthology*.

His most famous mathematical achievement is the invention of the sieve of primes (Fig. 13.2). It works as follows: All odd numbers, starting with three in the

Fig. 13.2 Sieve of Eratosthenes

desired number range (up to N) are filled into the sieve. First, all multiples of three are sieved, so 6, 9, 12, etc. The next smaller number remaining in the sieve is 5, so all multiples of 5 are sieved. This process is continued with 7, 11, 13, 17, … until the number $\sqrt{N}$ is exceeded. The numbers remaining in the sieve are the desired primes in the range of $\{3, .., N\}$. The number 2 was not considered a prime number. Since Eratosthenes' writings were lost, the sieve method was not known until Nikomachos' report (1, 13).

13.2 Eratosthenes as a Geographer

His most important achievement was the foundation of scientific geography. After Strabo's "Geographca", he collected distance measurements (from the records of the Egyptians and the Alexander campaigns), introduced latitudes and longitudes, and drew the corresponding maps.

Eratosthenes knew that the sun is vertical above Syene (today Aswan/Egypt) at the summer solstice (June 21st) (see Fig. 13.3). By measuring the sun's height in Alexandria on this day, the circumference of the earth $U = 2\pi R$ and thus the radius of the earth R could be calculated, given the known distance between Alexandria and Syene ($d = 5000$ stadia). He assumed that the two cities are on the same meridian, which is not exactly the case. Using the sun's height, he was able to measure the angle $\alpha = 7{,}2°$ that the sun's rays enclose in Alexandria with the plumb line. Assuming the sun ist very far away and the sun rays run parallel, the radius of the earth R can be determined to

$$\frac{d}{2\pi R} = \frac{\alpha}{360°} \Rightarrow R = \frac{d \cdot 360°}{2\pi\alpha} = 250.000\,St.$$

This is the result that Kleomedes handed down Theon of Smyrna and Strabo used the value 252,000 stadiums, possibly so that the number could be divided by 60. According to a conversion that can be found in Pliny's Natural History (XII, 53),

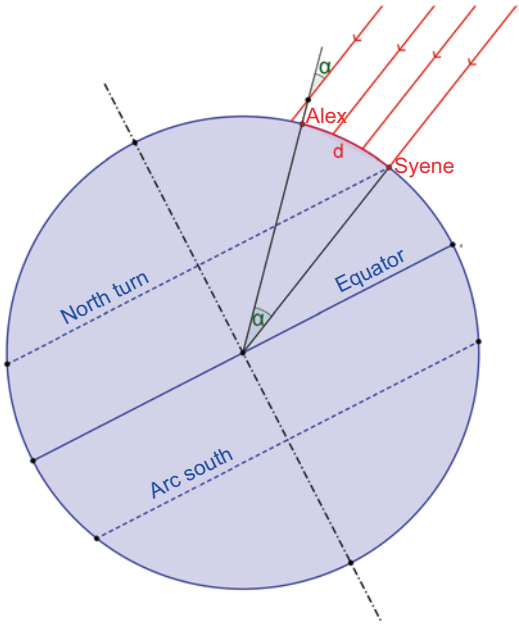

Fig. 13.3 Eratosthenes' measurement of the earth

the following applies: 1 stadium = 157.5 *m*. This gives an earth circumference of approximately 40,200 km, which is a very good value. According to the information of K. Ptolemaios, Eratosthenes calculated with an angular distance of the two tropics to $\frac{11}{83}$ of the full circle; that is 47°42′39″.

If you read the detailed description of the surveying by Kleomedes, you will get a more realistic picture of the measurement of angles. Kleomedes describes here specifically that for the measurement a horizontally set up semi-sphere, *Skaphe* ($\sigma\kappa\alpha\phi\acute{o}\varsigma$ = tub, ship's belly), called, was used (Fig. 13.4). T. Heath attributes the discovery of the skaphe to Aristarchos.

Kleomedes' report *De motu circulari* (I, 10) reads:

> That of Eratosthenes uses a geodesic method and seems some what obscure. He says that Syene and Alexandria lie below the same meridian. Since the meridians are great circles in the cosmos, those of the earth lying below them necessarily are also great circles. Thus this procedure shows that the size of the circle of the earth through Syene and Alexandria will be the same as the great circle of the earth.... He says, and this is so, that Syene lies below the summer tropical circle. Thus when the sun enters the Crab and makes the summer solstice, exactly at this meridian, by necessity the gnomons on the sundials are shadowless, since the sun lies exactly above. This is said to be [an area] 300 stadia in diameter. In Alexandria, however, at the same time, the gnomons on the sundials cast a shadow, since this city lies further north than Syene.... The arc drawn from the point of the shadow of the gnomon around to its base stands on this, and that from Syene to Alexandria stands on the center of the earth. The arcs are similar to each other, since they stand on equal angles. The ratio of the one in the base to its own circle is the same as the ratio of that drawn from Syene to Alexandria. The one in the basin is found to be a fiftieth part of its own circle. Necessarily, then, the distance from Syene to Alexandria must be a fiftieth part

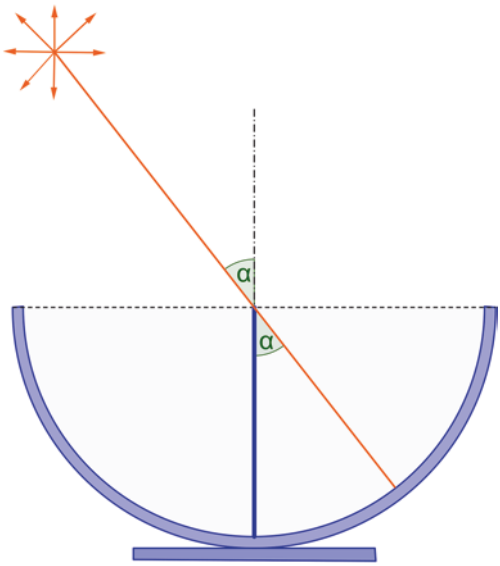

Fig. 13.4 Measurement of the sun's height by means of the skaphe

> of the great circle of the earth, and this is 5,000 stadia. Thus the entire circle is 250,000 stadia. This is Eratosthenes' procedure.

Pliny the Elder, who perished during the 79 A.D. eruption of Vesuvius while attempting a rescue, pays tribute to the surveying of the earth in his *Natural History* (II, 247) with the following words:

> Eratosthenes, who was deeply versed in all branches of learning and in this one more than in any other, has given the circumference of the whole earth as 252,000 stadia; and, so far as I can see, his opinion is shared by all. This is a bold statement, but one that has been won by such thorough investigation that one would be ashamed not to believe it.

It is often written that in ancient times the earth was considered to be flat. This is by no means true of Greek astronomy, since Aristotle demanded a spherical shape for *all* heavenly bodies. This is shown by the consideration of the famous *Atlas Farnese,* a Roman copy (circa 150 A.D.) of an older Greek statue carrying the celestical globe , (adorned with many star constellations) on his shoulders. The first book of the Middle Ages in which the flat-earth theory of the earth's orbit is taken up again is the encyclopedia of Isidore of Seville (636 posthumously published).

The idea of Eratosthenes was copied about 150 years later by *Poseidonos* from Apameis (now Syria). He tried to measure the culmination height of Sirius in Alexandria when the star appears on the horizon at the same time on Rhodes. He received the value 7.5° against the true value 5.25°, which leads to a larger error than with Eratosthenes. The resulting, too small value of the Earth's radius and a shortened quotation from Aristotle probably prompted Columbus to start his

Abb. 13.5 Spanish postage stamp with Aristotle quotation

circumnavigation. Columbus found in the book *Imago Mundi* by Pierre d'Ailly, the wrongly reproduced quotation from the book *Opus majus* (IV, 290) by Roger Bacon:

> Between the limits of Spain in the west and the shores of India in the east, the sea is small; it can be crossed in a few days with favorable winds.

Aristoteles (Abb. 13.5) had only written: *The sea is small between the boundaries of Spain in the west and the shores of India in the east.* The following sentence from Book V of *Naturales Questiones* by Seneca is: *The sea can be crossed in a few days with a favorable wind.*

According to the biography of Columbus' son, his father also knew the *Geography* by Strabo, which contains the following quotations from Eratosthenes and Poseidonos:

> If it were not for the size of the Atlantic Ocean, one could sail on the same parallel [latitude] from Iberia to India […]
>
> That one could, sailing east from the west, come to India after just as many stadiums [70,000]

Columbus was therefore probably not aware of the length of his journey when he sailed to "India". The oldest measurement of the earth comes from Dikaiarchos from Messina (around 300 BC), who was a geographer at the Lykeion. He observed that the constellation of the dragon in Lysimachia and the lion in Syene (approximately on the same meridian) appeared in the zenith at the same time. The angular distance of the two constellations he estimated at $\frac{1}{15}$ of the full circle. With a distance of 20,000 stadia of the two places, the circumference of the earth results in 300,000 stadia. This value was used by Archimedes in his writings.

The map of Eratosthenes (Fig. 13.6) shows Thule Island (probably the Orkney Islands) in the upper left, the Pillars of Hercules (= Gibraltar) on the left, the mouth of the Ganges on the right, and Taprobane Island (= Ceylon) in the lower

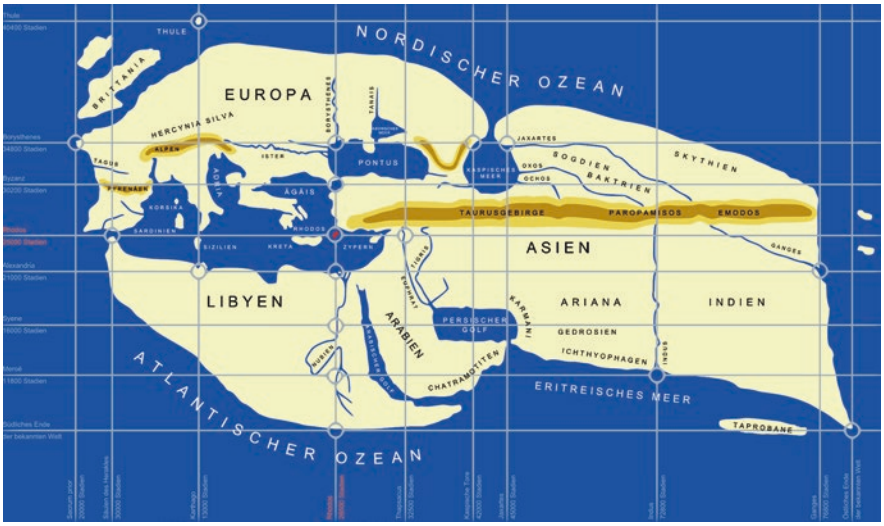

Fig. 13.6 Circle of the Earth according to Eratosthenes (Wikimedia Commons)

right. The depicted section includes the then known Oikumene (Greek οἰκουμένη) with an area of 38,000 stadia by 78,000 stadia. Eratosthenes laid out the main longitudinal and latitudinal lines through Rhodes; additional latitudes ran through cities whose width was known, such as Byzantium, Alexandria, Syene, and Meroë.

Eratosthenes' extensive "Geography", which consisted of three books, has not been preserved. However, its content and the polemical criticism of Hipparchus about it are largely reproduced by Strabo. In the first book of the work, Eratosthenes sketched the history of geography from the most ancient times. In doing so, he expressed criticism of the geographical information of the "unmistakable Homer", reported on the first geographical maps of Anaximander and Hecataeus, and defended the travel description of Pytheas, which was often ridiculed by his contemporaries.

The aforementioned astronomer Hipparchus accused his contemporary Eratosthenes of representing the hypothesis of the island-like shape of the *Oikumene* as an established fact in his work *Against the Geography of Eratosthenes*. Although Eratosthenes had recognized the importance of astronomical positional determination for geography, he had only gone halfway, since he also used length measurements from other sources, e.g. step counts of Alexander's army, not having precise astronomical measurements. Hipparchus had superimposed a triangular grid over the locations with known coordinates and found that the given distances were not consistent with his triangular calculation. However, this criticism was only partially justified, since there were not enough astronomical records from which the sought-after distances could be derived.

The work *Geography* by Strabo contains only fragments[2] of the mentioned work by Hipparchus, which comprises a total of 17 volumes. The fragments of Eratosthenes were collected by the same author[3] .

Eratosthenes divided the earth into the following zones:

Arctic circle	36°	25 200 stadia
Tropic of Cancer	30°	21 000 stadia
equator	48°	33 600 stadia
Tropic of Capricorn	30°	21 000 stadia
Antarctic circle	36°	25 200 stadia

According to a report by the famous, later living doctor Galenos from Pergamon (129-216 AD) in *Institutio logica*, Eratosthenes also dealt with astronomy. For example, he calculated the size and distance of the sun and moon, partial and total eclipses of the two celestial bodies and the dependence of the length of the day on season and latitude. He estimated the lunar distance to be 780 000 stadia or to the sun 804 000 000 stadia. The latter distance corresponds to about 86% of the astronomical unit with the conversion given above. Eratosthenes was also a great humanitarian. Strabo (Geography I,49), quotes extensively from Eratosthens' lost works: Now, towards the end of his treatise – after withholding praise from those who divide the whole multitude of mankind into two groups, namely, Greeks and Barbarians, and also from those who advised Alexander to treat the Greeks as friends but the Barbarians as enemies – Eratosthenes goes on to say that it would be better to make such divisions according to good qualities and bad qualities; for not only are many of the Greeks bad, but many of the Barbarians are refined – Indians and Arians, for example, and, further, Romans and Carthaginians, who carry on their governments so admirably. And this, he says, is the reason why Alexander, disregarding his advisers, welcomed as many as he could of the men of fair repute.

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[3] Berger H. (ed.): The geographical fragments of Eratosthenes, newly collected and discussed, Teubner 1880.

Conic Sections

14

The conic sections got their name from the fact that they can be generated as the intersection of a straight circular cone with a plane (Fig. 14.1). The following representation uses the method of surface constructions, which was known since Euclid. The terminology has been adopted from Apollonius.

The Ellipse

Let x be the abscissa of any point inside on the major axis; y the corresponding ordinate (Fig. 14.2). The ellipse is defined by the condition that the area of the rectangle $(x, 2p)$ is greater than the square on the ordinate y

$$y^2 < 2px$$

The parameter of the ellipse with the semi-axes a, b is defined by $p = \frac{b^2}{a}$. Since the ordinate square is less than the rectangle area, the figure was called an ellipse (ἔλλειψις = missing).

The Parabola

Let x be the abscissa of any point on the axis and y the corresponding ordinate. Then the parabola is defined by the fact that the area of the rectangle $(x, 2p)$ is equal to the square applied to the corresponding ordinate y

$$y^2 = 2px$$

The parameter of the parabola is defined by the distance of the focus in front of the baseline. Since the rectangle $(x, 2p)$ is equal in area to the square, the figure was called a parabola (παραβολή = side by side).

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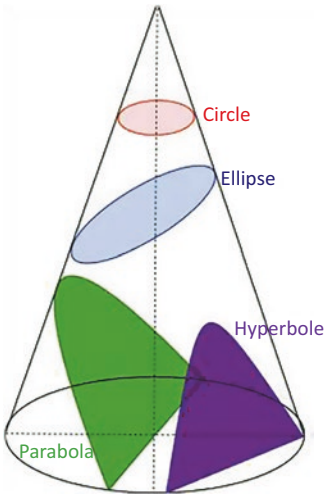


Fig. 14.1 Conic sections and their origin

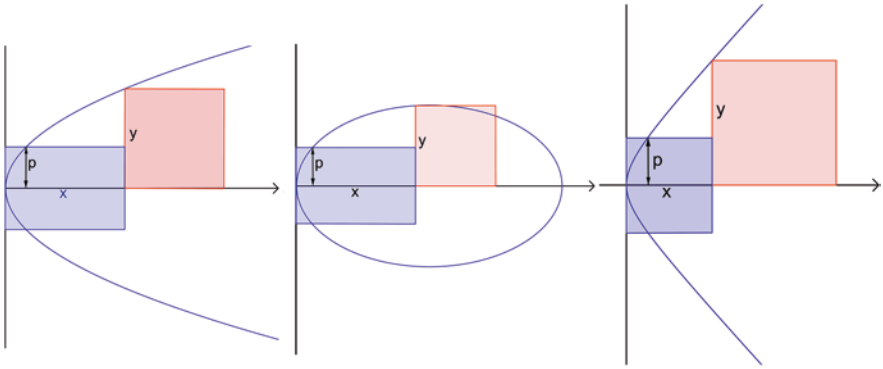


Fig. 14.2 Area application for the conic sections

The Hyperbola

Let x be the abscissa of any point on the axis and y be the corresponding ordinate. Then the hyperbola is defined by the fact that the area of the rectangle $(x, 2p)$ is less than the square applied to the ordinate

$$y^2 > 2px$$

The parameter of the hyperbola is defined by $p = \frac{b^2}{a}$,where the asymptotes are given by the pair of lines $y = \pm\frac{b}{a}x$. Since the ordinate square is greater than the rectangle area $2px$, the figure was called a hyperbola (ὑπερβολή = overshoot).

Transition to Coordinate Form

In a circle with radius r, for any point $(x; y)$ of a diameter, the altitude theorem

$$y^2 = x \cdot x_1; \quad x_1 = 2r - x$$

For the ellipse as an affine image of the circle, for any point (x; y) of the diameter $|AB|$ with a stretch factor $k = \frac{b}{a}$

$$y^2 = k^2 x \cdot x_1$$

(Fig. 14.3). If you set the origin of the coordinates in the center of the ellipse, then with $x \to x + a, x_1 \to a - x$

$$y^2 = k^2(x + a) \cdot (a - x) = k^2\left(a^2 - x^2\right)$$

The diameter AB was set equal to twice the semi-major axis $|AB| = 2a$. Substituting the stretch factor gives

$$y^2 = \left(\frac{b}{a}\right)^2 \left(a^2 - x^2\right)$$

Simplifying finally gives the standard form of the ellipse (center at the origin)

$$\begin{aligned} a^2y^2 = b^2\left(a^2 - x^2\right) &= b^2a^2 - b^2x^2 \\ b^2x^2 + a^2y^2 &= a^2b^2 \\ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= 1 \end{aligned}$$

For the hyperbola, the analogy is

$$y^2 = k^2(x + a) \cdot (x - a) = k^2\left(x^2 - a^2\right)$$

Again, the diameter AB is equal to $|AB| = 2a$. Substituting the stretch factor gives (Fig. 14.4):

$$y^2 = \left(\frac{b}{a}\right)^2 \left(x^2 - a^2\right)$$

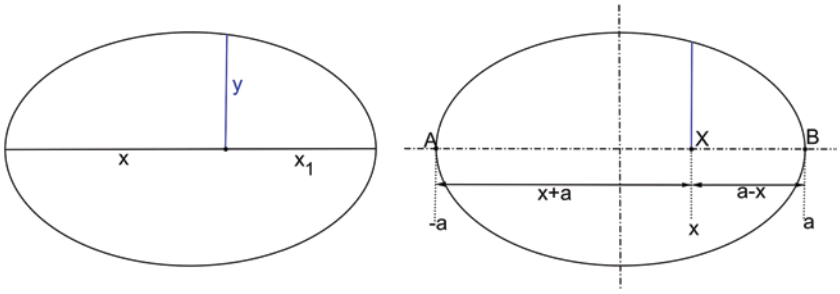


Fig. 14.3 Transition to coordinate form

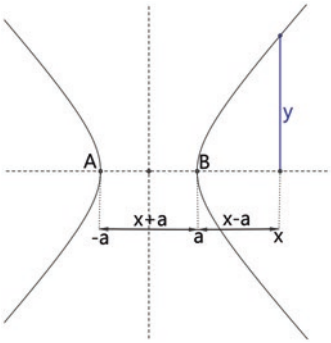


Fig. 14.4 Coordinate form of the hyperbola

This finally results in the standard form of the hyperbola.

$$a^2y^2 = b^2\left(x^2 - a^2\right) = b^2x^2 - b^2a^2$$

$$b^2x^2 - a^2y^2 = a^2b^2$$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

Representation in Greek Geometry

Of course, the Greek authors did not know any coordinate system, they always had to express the abscissae or the ordinates of the intersection points of the cone by suitable proportions. For the parabola, the proportion

$$\frac{|PQ|^2}{|RS|^2} = \frac{|AQ|}{|AS|}$$

It can also be found in Archimedes (*Quadrature of the Parabola,* 3) or Apollonius I, 21. For the ellipse and hyperbola, one needs the double proportion (Apollonius I, 21) (Fig. 14.5)

$$\frac{|PQ|^2}{|RS|^2} = \frac{|AQ|}{|AS|}\frac{|QB|}{|SB|}$$

The changed orientation of diameter AB has to be considered here.

Apollonius uses the following terms in his work (Fig. 14.6):

Term	Definition	Example
Chord	Connection between two points of the conic	BQ, CR, DS
Diameter	Line through the midpoints of all parallel chords	AZ
Vertex	Intersection of a diameter with a conic	A
Ordinate	Length of a half chord	XR, ZT
Axis	Line parallel to a diameter	

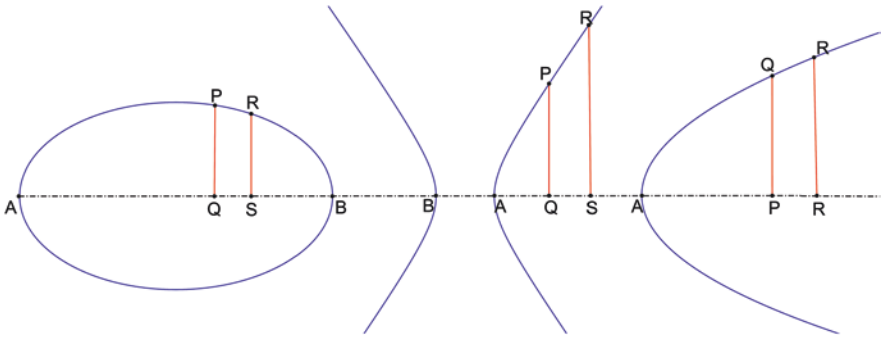
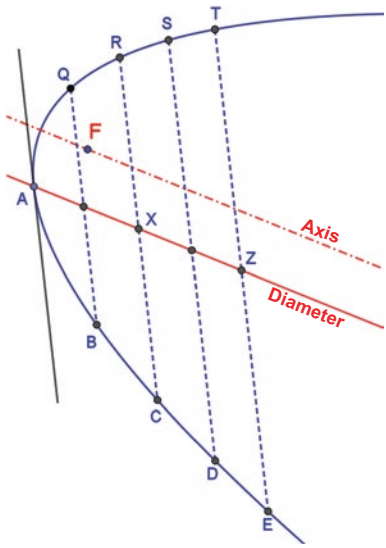


Fig. 14.5 Without coordinates, conic sections are calculated by proportions

Abb. 14.6 Figure to the table of the terms Apollonios



14.1 The Parabola

In addition to the definition according to Apollonius, the parabola can also be characterized by its focus or its guideline. For any point $P(x|y)$ on the parabola, the distance d to the focus F and to the directrix is equal to $d = |PF| = |PL|$. If we set the origin of the coordinate system in the vertex S, then according to the Pythagorean theorem (Fig. 14.7):

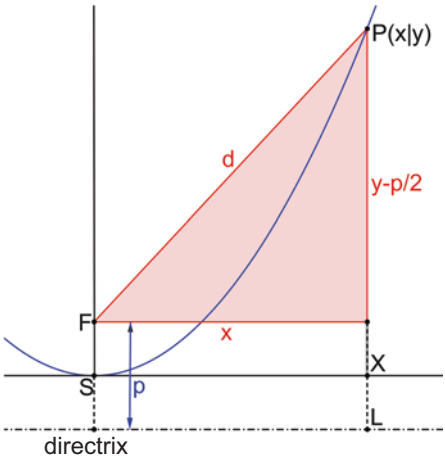


Fig. 14.7 Parabola by means of focus and directrix

$$\left(y+\frac{p}{2}\right)^2 = x^2 + \left(y-\frac{p}{2}\right)^2$$

$$y^2 + yp + \frac{p^2}{4} = x^2 + y^2 - yp + \frac{p^2}{4}$$

$$\Rightarrow x^2 = 2yp$$

$$\Rightarrow y = \frac{1}{2p}x^2$$

Interchange of the variables $x \leftrightarrows y$ yields the normal form of the parabola for *horizontal* axis

$$y^2 = 2px$$

Properties of the Parabola

1. Tangent at a point (Fig. 14.8):

At point P, the perpendicular is dropped to the axis; the foot point is Q. The point of intersection of the tangent at P with the axis is T. According to Apollonius II, 49, $|TS| = |SQ|$, where S is the vertex. Applying the length $|TS|$ on the symmetry axis gives the point T; the desired tangent is TP. TQ is called the subtangent of TP.

2. Tangent from a point P outside

A modern construction is shown in Fig. 14.9:

The circle is erected over the diameter PF; where F is the focus. The intersections of this circle with the vertical axis (in the apex S) are A and C. The lines AP and CP are the tangents with the points of contact B_1, B_2.

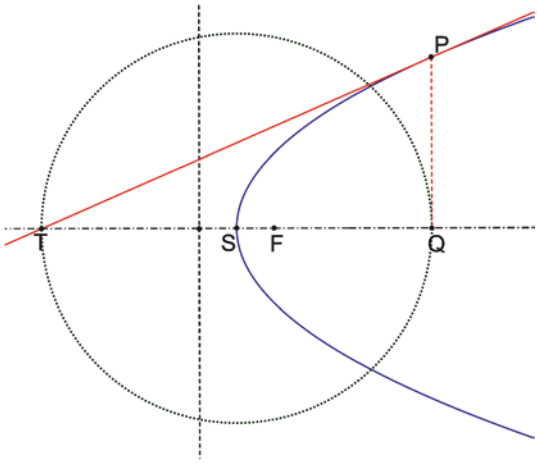


Fig. 14.8 Tangent construction

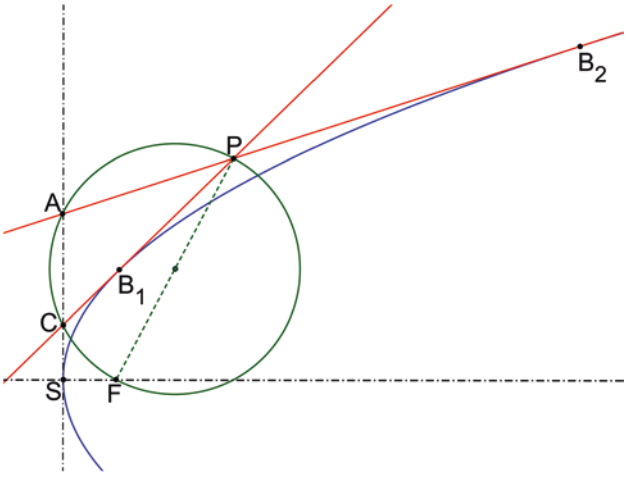


Fig. 14.9 Tangent construction from point P outside

3. Focus property (Fig. 14.10):

At a point of contact, the connecting line to the focus encloses the same angle with the tangent as a parallel axis line. The *reflection law* applies: A line parallel to the axis is broken in a point P towards the focus J. Kepler already suspected that the parabola has a second focal point, which is located in the infinite.

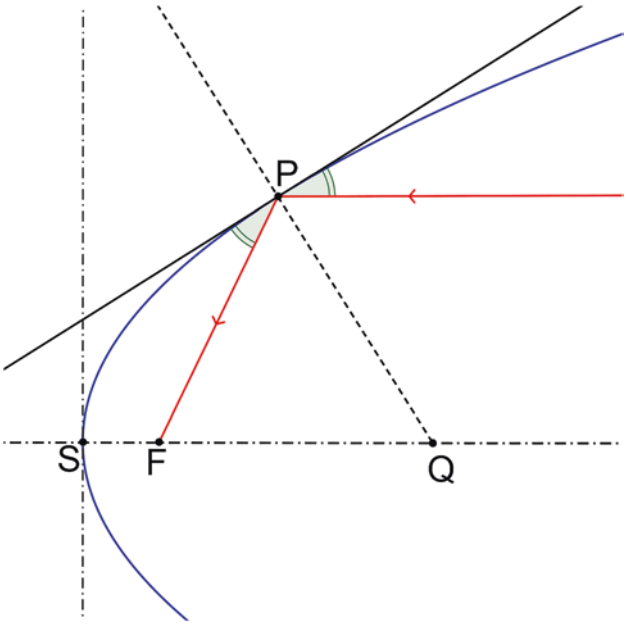


Fig. 14.10 Reflection law at a parabola

4. Another tangent method (Fig. 14.11):

From the properties (1) and (3), a new construction for the tangent in a parabola point P results. Let G be the perpendicular projection of P onto the directrix.

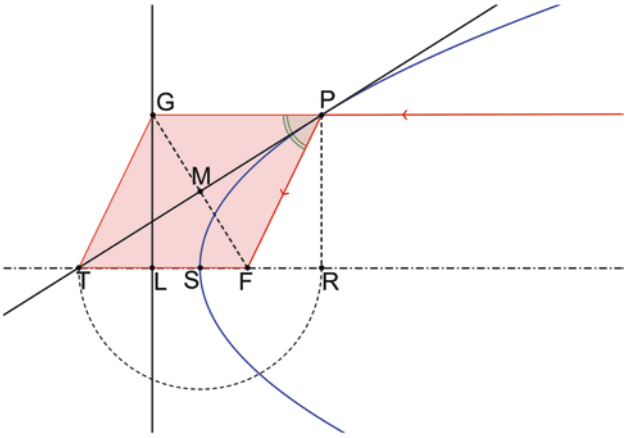


Fig. 14.11 Another tangent construction

The tangent in the point P is the bisector of the angle ∢GPF, where F is the focus of the parabola.

Explanation: According to the definition of the parabola, $|PG| = |PF|$. Thus, the triangle GFP is isosceles. According to (3), the angle at P is bisected by the tangent PM or PT. Since the angle bisector in the isosceles triangle is also the median, M bisects segment GF. Since the vertex S bisects LF, MS is the median of △GLF. If R is the foot point of P on the axis, then △TSM and △TRP are similar. Since, according to Apollonius, S is the midpoint of TR, M midpoint of PT. This shows that GTFP is a rhombus.

5. Parabola segment (according to Archimedes)

Let PQ be the chord of the parabola and AB the parallel tangent at the point of contact C. According to Archimedes, the area of the parabolic segment $\frac{4}{3}$of the inscribed triangle △CPQ. The parallelogram AGQB is twice as large as the triangle. Thus: The area of the parabolic segment is $\frac{2}{3}$of the parallelogram area AGQB (Fig. 14.12)

6. Pointwise construction of the parabola

According to Ibrahim ibn Sinan (909–946), a grandson of Thabit ibn Qurra, a parabola can be constructed pointwise as follows. First, a circle with radius a is drawn with the center on the negative x-axis, which touches the y-axis. If you are looking for the parabola point to the abscissa x, then the circle with the center $\left(\frac{-2a+x}{2}; 0\right)$ is drawn through the point $(-2a; 0)$. At each intersection point with the x- and y-axis, perpendicular line is erected, the cut points provide the points of the parabola (Fig. 14.13).

Reasoning: The circle with center $(x,0)$ has the radius $r = a + \frac{x}{2}$; The cutting points on the axes forms a right-angled triangle with the sides $(r; r - x; y)$. According to Pythagoras, this is

$$\left(a + \frac{x}{2}\right)^2 = \left(a - \frac{x}{2}\right)^2 + y^2 \Rightarrow y^2 = 2ax$$

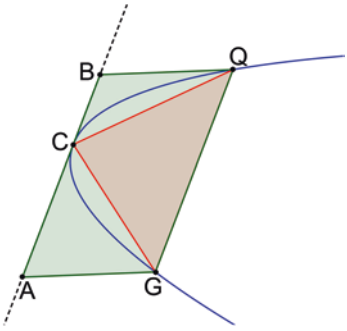


Fig. 14.12 Parabolic segment with inscribed triangle

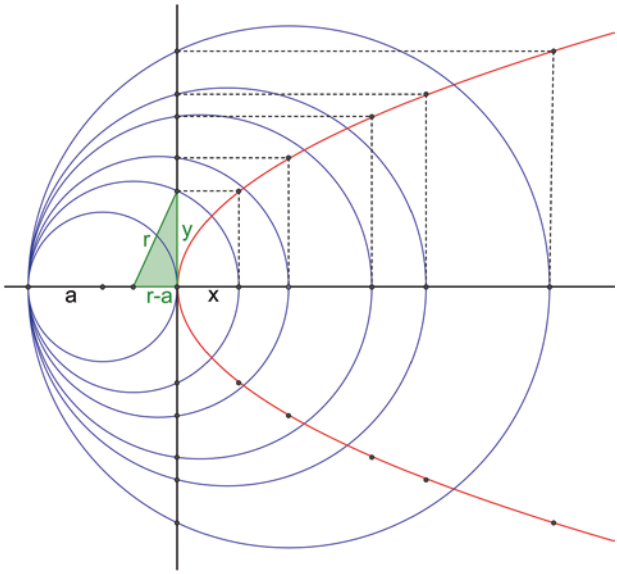


Fig. 14.13 Parabola construction according to ibn Sinan

The cut of each pair of perpendiculars actually provides a point of the parabola.

More Theorems About Parabola Tangents

7. If three parabola tangents intersect at the points of contact B_1, B_2, B_3 at points *P, Q, R*, then the tangent sections are divided in the same proportion. It holds (cf. Fig. 14.14):

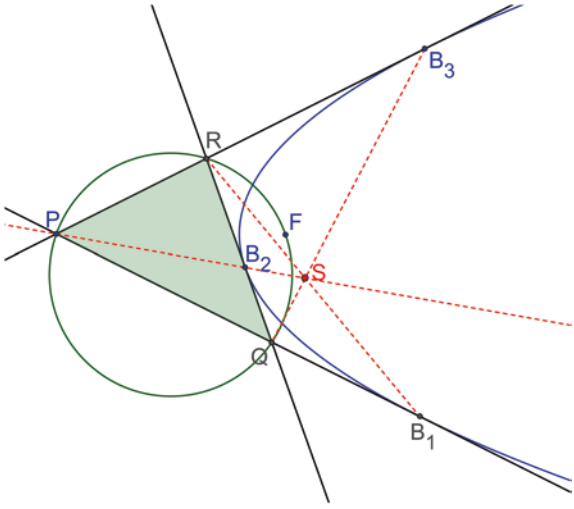


Fig. 14.14 Tangent triangle at parabola

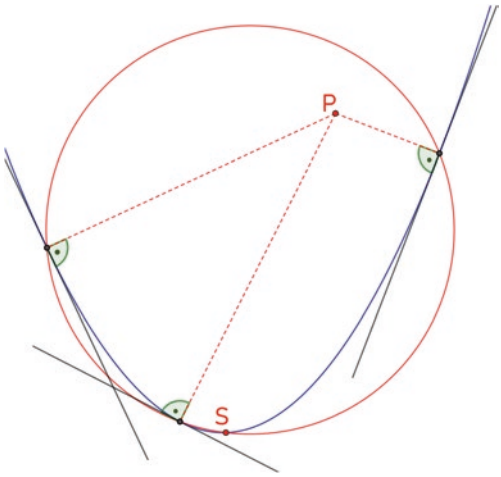


Fig. 14.15 3 Tangenten an Parabel

$$\frac{|B_3R|}{|RP|} = \frac{|PQ|}{|QB_1|} = \frac{|RB_2|}{|B_2Q|}$$

(Apollonius III, 41)

8. If three parabola tangents intersect at the points B_1, B_2, B_3 in pairs, then the points P, Q, R of intersection form a triangle, on whose circumference lies the parabola focus F (theorem of Lambert).
9. If one connects the points of intersection of two tangents with the point of intersection of the third tangent, then these connecting lines intersect in one point S *(theorem of* Steiner*).*
10. If one drops three lines from a point inside a parabola onto the curve, then the three points of intersection of the lines and the vertex lie on a circle (theorem of Joachimstahl) (Fig. 14.15).

14.2 The Ellipse

In addition to the definition according to Apollonius, the ellipse can also be defined as a geometric locus. An ellipse is the set of all points P that have the same sum of distances from two fixed points F_1, F_2

$$|PF_1| + |PF_2| = 2a$$

In this case, 2a is the large diameter of the ellipse.

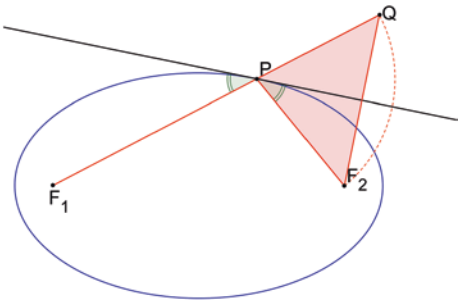


Fig. 14.16 Tangent to ellipse

Properties of the Ellipse

1. Tangent in one point (Fig. 14.16)

The point P is connected with the two foci F_1, F_2. The line F_1P is produced by $|F_2P|$ to the point Q. The altitude of the isosceles triangle ΔPF_2Q gives the desired tangent in the point P.

2. Linear excentricity

Since focus and center only coincide for circular ellipses, J. Kepler introduced the linear eccentricity e as a measure of the distance between focus and center. According to the Pythagorean theorem (Fig. 14.17):

$$a^2 = b^2 + e^2 \Rightarrow e = \sqrt{a^2 - b^2}$$

For many applications, such as in astronomy, the linear eccentricity is set in relation to the major axis; the *numerical eccentricity* then results

$$\varepsilon = \frac{e}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

3. Ellipse parameters

Half the length of the perpendicular secant through one focus is called the ellipse parameter p. According to the Pythagorean theorem (Fig. 14.18):

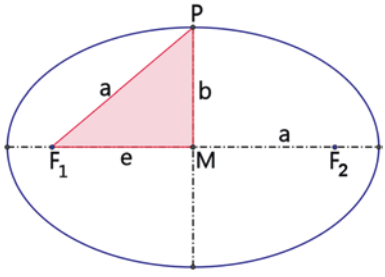


Fig. 14.17 Definition of eccentricity

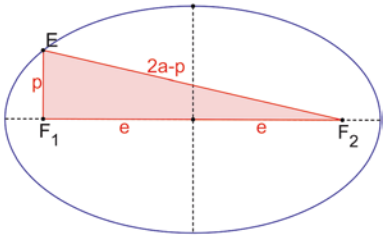


Fig. 14.18 Ellipse parameters

$$p^2 + (2e)^2 = (2a - p)^2 \Rightarrow a^2 - ap = e^2$$
$$\Rightarrow b^2 = ap \Rightarrow p = \frac{b^2}{a}$$

4. Area of the ellipse

With the help of the affinity $y' \to y\frac{b}{a}$ (here a stretching in y-direction), Archimedes derives the area formula in § 5 of his book about rotation-symmetric solids

$$\mathcal{F}_{ellipse} = \frac{b}{a}\mathcal{F}_{circle} = \frac{b}{a} \cdot \pi a^2 = \pi ab$$

More Properties of the Ellipse

5. Construction of C.F. Gauß: With the help of two secants, the polar *SR* of a point P is determined; this also allows to find the tangents. Such relationships are the subject of projective geometry (Fig. 14.19).
6. Ellipse as a projection of a circle onto a plane (Fig. 14.20):
7. Ellipse as the image of a circle in a 2-point perspective (Fig. 14.21):
8. Conjugate diameters (Fig. 14.22):

Conjugate diameters divide the ellipse into 4 sectors of equal areas $\mathcal{F} = \frac{1}{4}\pi ab$. The parallelograms, too, in which the ellipses are inscribed, are equal in area with $\mathcal{F} = 4ab$; they are also called conjugate parallelograms.

14.3 Hyperbola

In addition to the definition according to Apollonius, the hyperbola can also be defined as a geometric locus. A hyperbola is the set of all points P that have the same difference of distances from two fixed points F_1, F_2.

$$|PF_1 - PF_2| = 2a$$

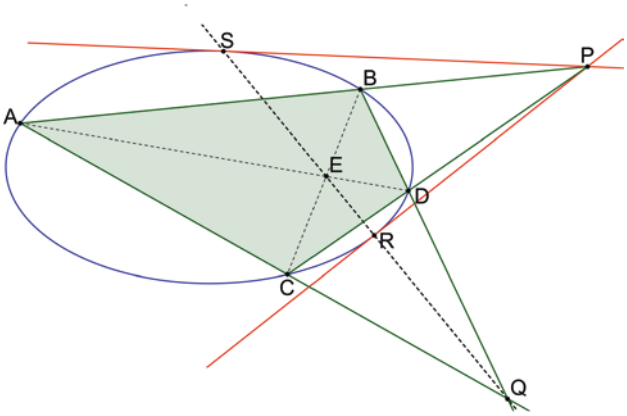


Fig. 14.19 Relationship between tangents and polar

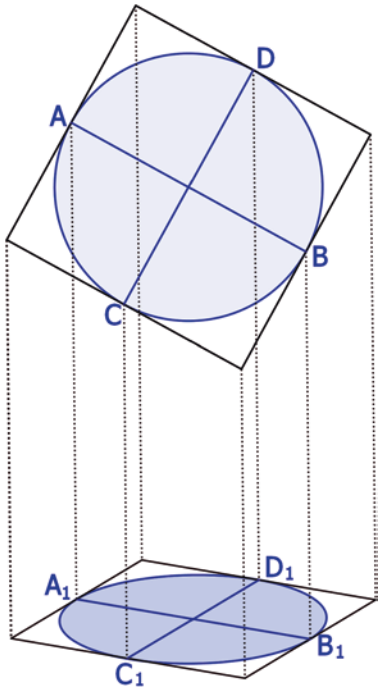


Fig. 14.20 Projection of a circle into a plane

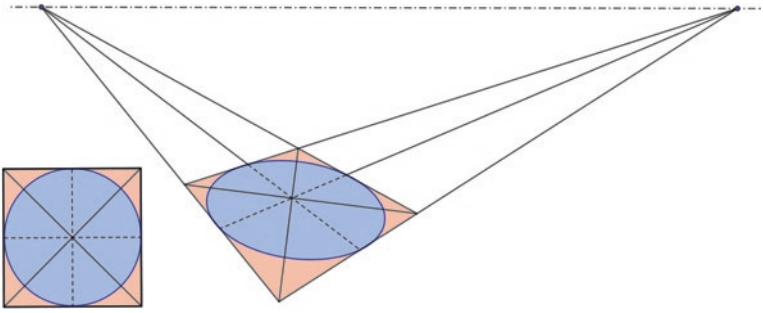


Fig. 14.21 2-point perspective of a circle

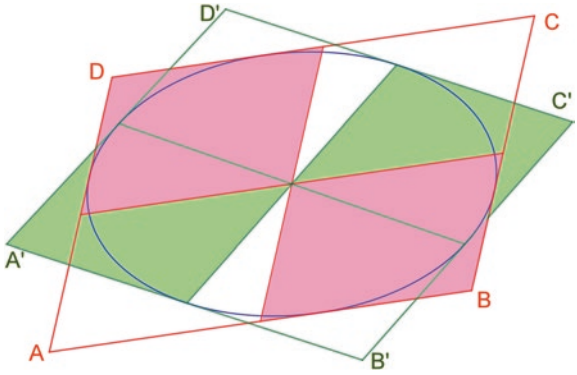


Fig. 14.22 Conjugate diameters

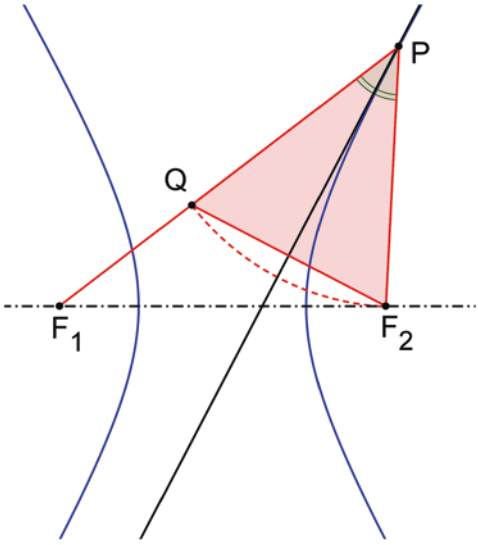


Fig. 14.23 Tangent to hyperbola

Properties of the Hyperbola

1. Tangent in one point (Fig. 14.23).

Point P is connected to the two focuses F_1, F_2. The line F_1P is shortened by the distance $|F_2P|$ to point Q. The median of the isosceles triangle $\triangle PF_2Q$ is the desired tangent at point P.

2. Axes and asymptotes

The two axis sections 2a, 2b are sides of a rectangle, whose diagonals determine the direction of the asymptotes. The equations of the asymptotes are (Fig. 14.24):

$$y = \pm\frac{b}{a}x$$

3. Linear eccentricity

Since the center and focus are different, the linear eccentricity e is used as a measure of the distance between the center and focus. According to the Pythagorean theorem,

$$e^2 = a^2 + b^2 \Rightarrow e = \sqrt{a^2 + b^2}$$

4. Hyperbola parameter

The half length of the secant perpendicular to the axis through a focus is called the hyperbola parameter p. It applies as with the ellipse $p = \frac{b^2}{a}$.

5. Each tangent to the hyperbola intersects with the asymptote pair a constant area (Apollonius II,43): $\mathcal{F} = ab$ (Fig. 14.25).
6. Each section of a tangent between the asymptotes is halved by the point of contact (Apollonius II, 3).

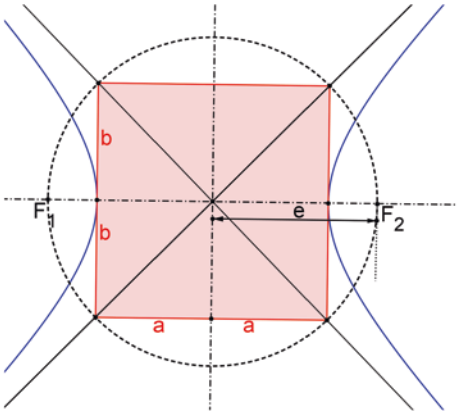


Fig. 14.24 Asymptotes and eccentricity

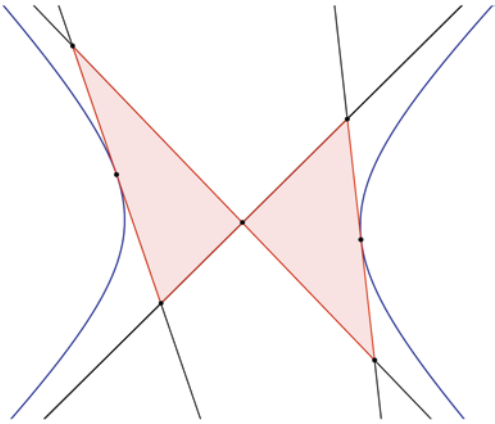


Fig. 14.25 Apollonius II,43

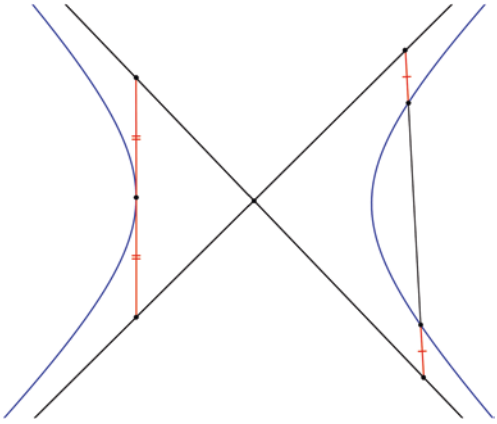


Fig. 14.26 Apollonius II,8

7. The sections between hyperbola and asymptotes, which are generated by the intersection with a secant, are congruent (Apollonius II, 8) (Fig. 14.26):

8. Let AD be a secant of a hyperbola, with AB and CD being the sections cut off by the curve and the asymptotes. Through C and B, the parallels to the asymptotes are drawn. The rectangular areas formed by the asymptotes and the parallels and the perpendiculars in C and B are equal in area. With the notation of Fig. 14.27 we have

$$\mathcal{F}(\text{MKBH}) = \mathcal{F}(\text{MECG})$$

Since the triangles $\triangle$KAB, $\triangle$EFW and $\triangle$GCD are congruent in pairs, the following parallelograms are also equal in area

$$\mathcal{F}(\text{WEAB}) = \mathcal{F}(\text{WECD})$$

Conclusion from (Apollonius II,8):

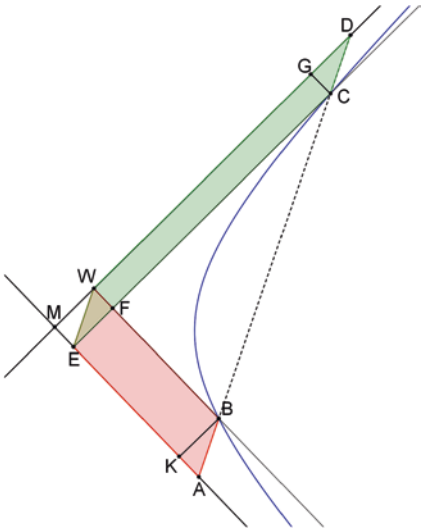


Fig. 14.27 parallelograms of equal areas

9. Rectangular hyperbola

If the two semi-axes are equal ($a = b$), the hyperbola is called rectangular. The asymptotes are then perpendicular to each other.

If you inscribe a triangle ABC in an rectangular hyperbola, the orthocenter H will also lie on the circumhyperbola (Fig. 14.28).

10. Intersection of an rectangular hyperbola

If a rectangular hyperbola intersects a circle, then there are generally 4 intersection points A, B, C, D. One of the points (in the figure point D) is the mirror point of orthocenter H of the triangle $\triangle$ABC, reflected at the center Z of the hyperbola.

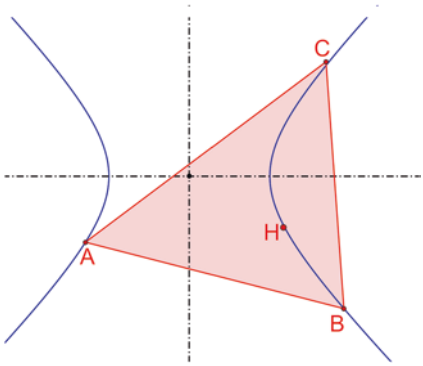


Fig. 14.28 Orthocenter of a triangle with circumhyperbola

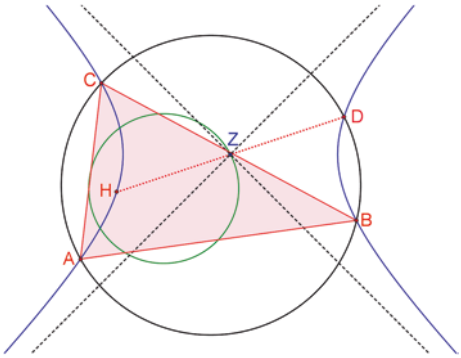


Fig. 14.29 Feuerbach circle of the inscribed triangle

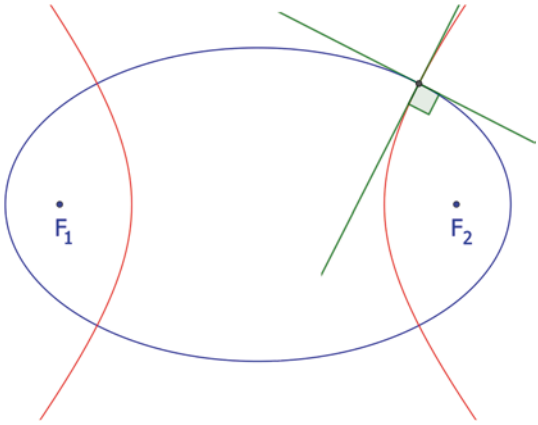


Fig. 14.30 Confocal ellipses and hyperbolas

The center Z of the hyperbola therefore lies on the Feuerbach circle of the triangle △ABC (Fig. 14.29).

11. Confocal ellipses and hyperbolas intersect orthogonally (Fig. 14.30).

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Apollonius of Perga, Heath T.L. (Ed.): Treatise on conic sections. Carruthers Press, Providence (2010)

Czwalina, A. (Ed.): Die Kegelschnitte des Apollonios. Wissenschaftliche Buchgesellschaft. De Gruyter, Darmstadt 1967[1], (2019)[2]

Zeuthen, H.G.: Die Lehre von den Kegelschnitten im Altertum. Höst & Sohn, Kopenhagen (1886)

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Apollonius of Perga

15

Very little is known about the life of Apollonius (Απολλώνιος) of Perga/Perge. As his name suggests, he was born in Perga (now Murtina/Turkey) during the reign of Ptolemy III Euergertes. This is reported by *Heraclius,* the author of a (lost) biography of Archimedes, which in turn is based on writings by Eutocius. He is not to be confused with his namesake Apollonius of Rhodes (the famous poet) or Apollonius of Tyana (the philosopher). According to Pappus he studied mathematics in Alexandria at the school founded by Euclid. During the reign of Ptolemy IV Philopator (221–203) he also worked as an astronomer in Alexandria, before moving to Pergamon later. When he died, he was, according to the words of Geminus of Rhodes, so famous that he was given the name *Great Geometer*. In the list of great mathematicians in Book I of *Architectura* by Vitruvius he is even mentioned before Archimedes! His popularity is also reflected in the large number of commentators: Pappus, Hypatia, Serenus and Eutocius.

We can find some information in the prefaces of his books on *Conica* (κωνικά = conic sections). In the preface of Book I, Apollonius writes:

> Apollonius greets Eudemus,
>
> If you are in good health and all other circumstances are as you wish, it is good. […] During the time we spent together in Pergamon, I noticed your great interest in becoming acquainted with my investigations. Therefore, I am sending you the first book that I have revised in the meantime […] As you will remember, I told you at that time that I was working on the studies of this subject [conic sections] at the suggestion of Naucrates, the geometer, at the time when he visited and stayed with me in Alexandria. When I had finished the work in 8 books, I hastily handed them over to him because he was about to sail. They [the books] were not carefully enough reviewed at that time; in fact, I had written everything down as it occurred to me and postponed the final revision.

The first three books are dedicated to Eudemus from Pergamon. In the preface of Book II, he mentions that he is sending his son Apollonius a copy of the book to him in Pergamon. At the same time, he allows him to pass on a copy of the book

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_15

to the geometer Philonides. As we know from other sources, Philonides was later active at the court of the Seleucid king Antiochus IV, who ruled from 175–163.

Apollonios dedicated books IV–VIII to a certain Attalos, whom we can assume to be identical with King Attalos I (reign 241–197 BC) of Pergamon, although he avoids the title *King* in his address. He writes at the beginning of book IV

> Apollonios greets Attalos:
>
> Some time ago I explained and sent to Eudemos of Pergamon my first three books of the Conica, which I have summarized in eight books; but he is now deceased. Therefore, I have decided to dedicate the remaining books to you, since it is your desire to possess these books.

If Attalos is the mentioned king, then Apollonios can be dated to 260–195 BC; this would make him a generation younger than Archimedes. G.J. Toomer[1] believes it is impossible that King Attalos is not addressed by his title *Basileus* (βασιλεύσ). However, we also know an Attalus from Rhodes, who was not a contemporary of Apollonios. Thus, the identity of Attalos remains unclear.

The first four books are, as already mentioned, preserved through the comments of *Eutokios* and therefore transmitted in Greek. The later books IV–VII were translated by Banū Mūsā and are only preserved in Arabic. The famous astronomer Edmond Halley was able to reconstruct book VIII to a large extent from the numerous comments of Pappus.

According to their own statements, books I–IV contain elementary basics of conic section theory, which, according to the opinion of Pappos, largely go back to the lost treatise *Conic Sections* by Euclid. Here he defines the conic sections by comparison of areas; unlike his predecessor Menaichmos, who generated them by plane sections of cones with acute, right-angled and obtuse angles. The table provides an overview of the content of the *Conica* books.

I	Generation of the conic section and the circular cone
II	Axes and diameters of the conic sections
III	Transversals of the conic sections, theory of the pole and polar, focal point of the ellipse and hyperbola
IV	Investigation of the intersection of conic sections with circles
V	Theory of the normals and subnormals, shortest and longest connection with a point outside of the cone and the conic section
VI	Investigation of equal and similar conic sections
VII	Statements about special properties of conjugate diameters
VIII	Special construction tasks for conic sections

[1] Toomer G.J., Apollonius of Perga, *Dictionary of Scientific Biography* I (1970), p. 179.

Other works by Apollonius mentioned by Pappus

- De Rationis Sectione *(Sections with Ratio)*
- De Spatii Sectione *(Sections of Plane Areas)*
- De Sectione Determinata *(Determination of Sections)*
- De Tactionibus *(On Contacts)*
- De Inclinationibus *(On Inclinations)*
- De Locis Planis *(Geometric Loci of the Plane).*

Works reported by other authors:

- On the Burning Glass (Proof that a spherical mirror does not have an exact focal point)
- Improved calculation of π in comparison to Archimedes
- Improved and simplified version of Book X of the Elements
- On the comparison of Dodecahedron and Icosahedron

The aforementioned writing was edited by Hypsikles (of Alexandria) and added as Book XIV to the *Elements*. It contains the statement that if icosahedrons and dodecahedrons are inscribed in the same sphere, the surfaces of these solids are also inscribed in the same circle. Furthermore, it is claimed that the surfaces or volumes of these solids relate to each other as the edge of the cube to that of the icosahedron.

The *Conica* had great importance in the history of mathematics up to the beginning of the modern era. By his calculations using proportions, Apollonius was able to anticipate results that could only be treated analytically after the introduction of coordinate geometry by Fermat and Descartes. After it was translated into Latin by F. Commandino in 1599, J. Kepler was later able to use it for his astronomical calculations. Numerous theorems from the Conica were used and newly proven by I. Newton[2] in his *Principia*, in particular in section III "On the motion of bodies in eccentric conic sections". Newton encouraged translations of Apollonius into English: I. Barrow (1675) and Edmond Halley (1710). His epoch-making work *Geométrié* begins with the *Four-Line-Problem* of Apollonius, handed down by Pappos.

Book I–III of the *Conics* was newly edited by Dana Densmore[3]. Book IV was edited by Michael N. Fried[4]. Books V–VII of the Conica were published by G.J. Toomer[5]. The German edition of the Conics by Arthur Czwalina[6] was reprinted.

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[5] Toomer G.J. (ed.): Apollonius Conics Books V to VII, The Arabic Translation of the lost Greek Original in the Version of the Banū Mūsā, Volume I+II, Springer 1990.

[6] Czwalina A. (ed.): Die Kegelschnitte des Apollonios, Wissenschaftliche Buchgesellschaft 1967[1], De Gruyter 2019[2].

15.1 From Book III of the Conica

Theorem III, 42 Given a central cone. If straight lines are drawn from the vertices of a diameter AB parallel to an ordinate and any other tangent, it will cut off from them straight lines containing a rectangle equal the forth part of the figure to the same diameter. It holds $|AC||BD| = \frac{1}{4}|AB|p$. p is the parameter correponding to diameter AB. See Czwalina (p. 165), Densmore (p. 247). (Fig. 15.1a, b).

Theorem III, 44
If you draw two tangents to a hyperbola from a point, then the connecting lines of the points of intersection with the asymptotes and the point of tangency are parallel (Fig. 15.2).

Theorem III, 45
In the vertices of a central conic section, the tangent are erected and meet any other tangent. Connecting these points of intersection in pairs with the foci, the connecting lines form a right angle in each case. In Fig. 15.3a, b it holds: $\measuredangle QER = \measuredangle QFR = 90°$.

Theorem III, 48
The tangent at a point of a central conic intersects the lines connecting the foci at angles that are congruent (Fig. 15.4).

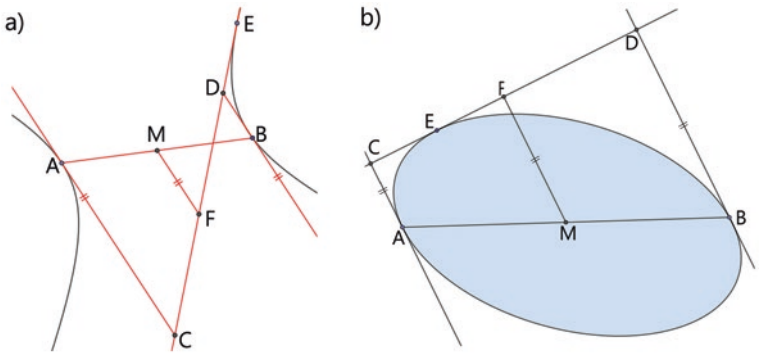

Fig. 15.1 Apollonius III, 42 for hyperbola and ellipse

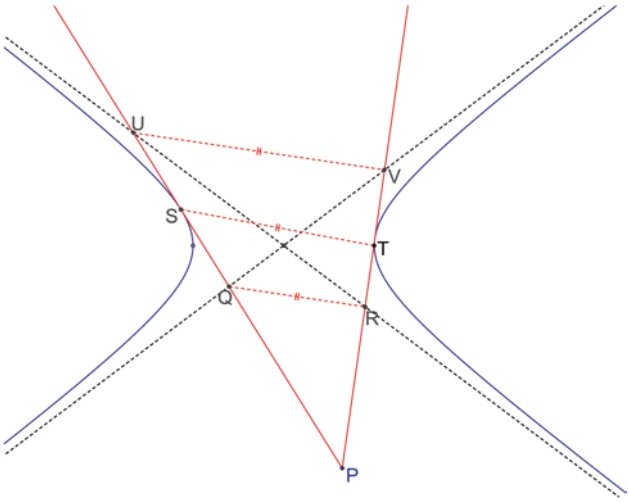

Fig. 15.2 Apollonius III, 44

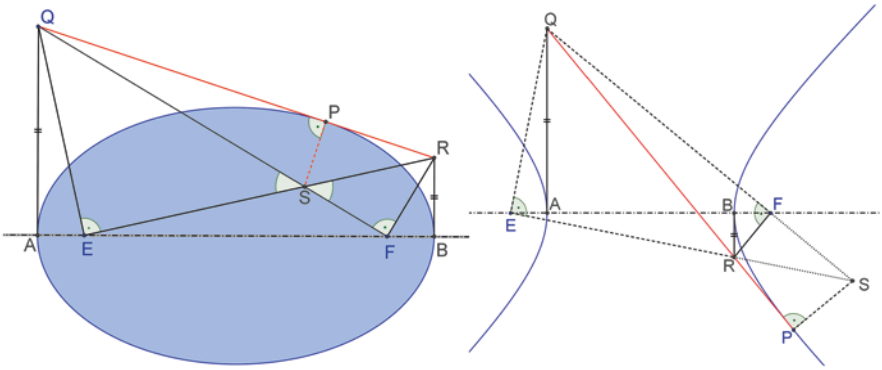

Fig. 15.3 Apollonius III, 45 for ellipse and hyperbola

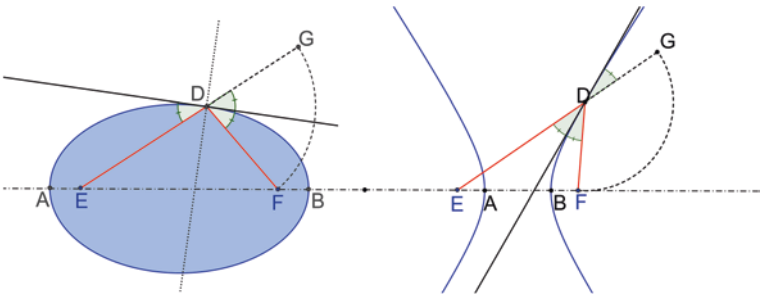

Fig. 15.4 Apollonius III, 48 for ellipse and hyperbola

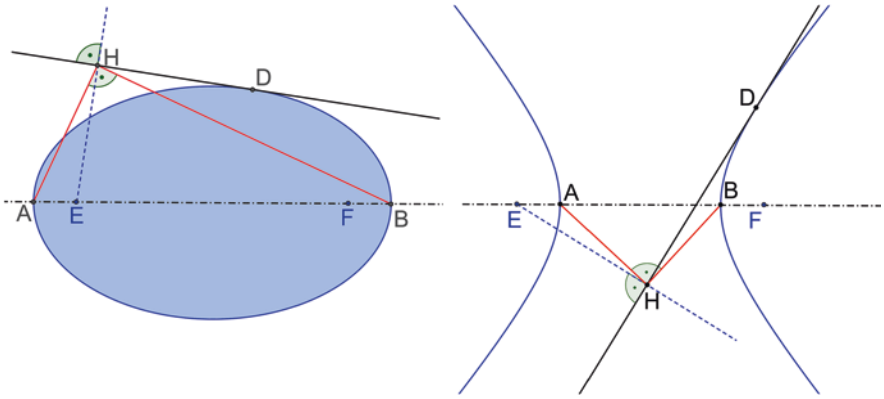

Fig. 15.5 Apollonius III, 49 for ellipse and hyperbola

Theorem III, 49
The foot of the perpendicular from a focus (of a central conic section) to a tangent intersects the lines connecting the vertices (of the major axis) at a right angle (Fig. 15.5).

Theorem III, 50
If you connect one focus (of a central conic section) with the one point of a tangent, then the tangent section of the parallel through the center is equal to the major semi-axis (Fig. 15.6).

Here it applies: $|CH| = \frac{1}{2}|AB|$.

Theorem III, 51+52
The sum of the distances of a point from the foci is equal to the diameter in the case of an ellipse; in the case of a hyperbola, this is the difference of the distances.

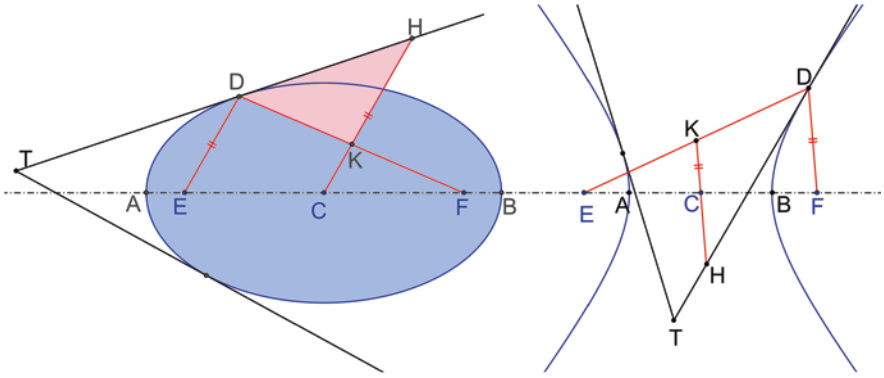

Fig. 15.6 Apollonios III, 50–52 for ellipse and hyperbola

To be shown is: $|ED| + |DF| = |AB| = 2a$ or $||ED| - |DF|| = |AB| = 2a$.

Proof in the ellipse case.

Given is the ellipse with the diameter AB and the focal points E, F. It is to be shown $|ED| + |DF| = |AB|$. As auxiliary lines the tangent through D, the parallel through the center point C and the line DF are drawn. With the intersection point H of the tangent it holds $CH \parallel ED$.

According to Theorem III, 48 the angles $\measuredangle TDE$ and $\measuredangle FDH$ are congruent. Because of the parallelism $CH \parallel ED$ also the angle of inclination $\measuredangle SHC$ is congruent. Thus it holds $\measuredangle FDH = \measuredangle SHC$; the ΔDKH is thus isosceles with $|DK| = |KH|$ Since the center point C of the ellipse is also the center between the focal points E, F it holds according to the quadrilateral theorem $|CK| = \frac{1}{2}|ED|$ and $|DK| = |KF|$ In total it follows

$$|ED| + |DF| = 2|CK| + 2|DK| = 2(|CK| + |KH|) = 2|CH| = |AB|$$

The last step follows from Theorem III, 50.

Theorem III, 53

Given are the parallel tangents at the endpoints A, B of a diameter of a central conic section. Through the point C of the conic section, the lines AC and BC are drawn. The intersections with the tangents generate the tangent segments t_1, t_2. It holds: The product of the tangent segments is equal to the product of the diameter and the parameter p corresponding to diameter AB $|AB|p$ (Czwalina, p. 174):

$$t_1 \cdot t_2 = |AB| \cdot p$$

Figure 15.7 only shows the case of the ellipse.

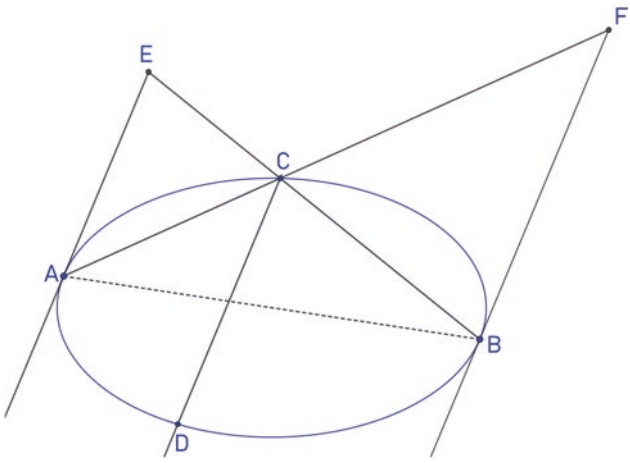

Fig. 15.7 Apollonius III, 53 for ellipse

15.2 Apollonius' Theorem

A well-known geometrical construction task is the so-called Apollonius circle; it is the geometrical locus of all points P that have the distance ratio $|AP| : |BP| = k(k \neq 1)$ from two given points A, B (Fig. 15.8). In his book *Meteorologica*, Aristotle already shows knowledge of this geometrical locus. Simplicius' assignment of the circle to Apollonius is therefore incorrect.

Proof: The coordinates of the points are $A(-a|0)$ and $B(a|0)$. respectively. The origin of the coordinate system can be placed in the midpoint of the line segment AB. It then follows that

$$|AP|^2 = (a+x)^2 + y^2$$

$$|BP|^2 = (a-x)^2 + y^2$$

Substituting into the distance condition yields

$$|AP|^2 = k|BP|^2 \Rightarrow (a+x)^2 + y^2 = k\left[(a-x)^2 + y^2\right]$$

Simplifying and sorting shows

$$a^2 + 2ax + x^2 + y^2 = ka^2 - 2kax + kx^2 + ky^2 \tag{15.1}$$

$$(k-1)x^2 + (k-1)y^2 - 2ax(k+1) = a^2(1-k)$$

Dividing by $(k-1) \neq 0$ yields the circle equation

$$x^2 + y^2 - 2ax\frac{k+1}{k-1} = -a^2$$

Quadratic supplement shows the circle

$$\left[x - a\frac{k+1}{k-1}\right]^2 + y^2 = a^2\frac{4k}{(k-1)^2}$$

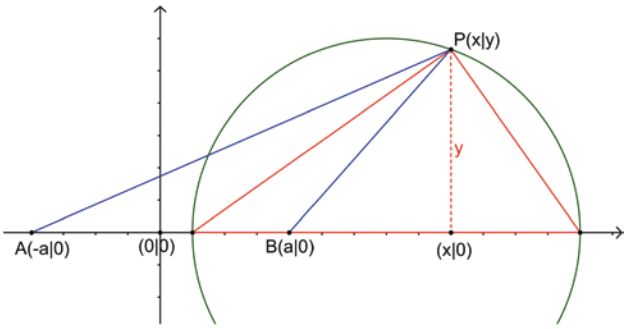

Fig. 15.8 On the derivation of the Apollonios circle

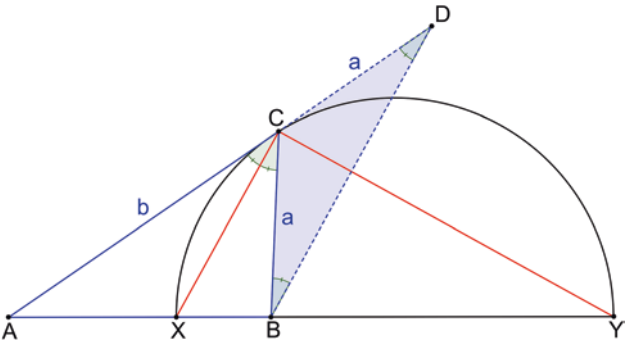

Fig. 15.9 Exterior and interior angle bisector

For limes $k \to 1$, we get from (15.1) the line $x = 0$, that is, the y-axis as the geometric locus; this is precisely the perpendicular bisector of AB.

According to Euclid (VI, 3), the angle bisector of an interior angle divides the opposite side in the ratio of the adjacent sides. In the triangle ABC, therefore, the bisector of the angle at C divides the opposite side AB in the ratio $\frac{b}{a}$ The line CY is the bisector of the exterior angle at C (Fig. 15.9).

It is to be shown that X is the point of division of AB. Producing side AC and intersecting to its parallel through B, we get the point D. Because of the parallelism, both the angle $\measuredangle CBD$ and angle $\measuredangle CDB$ are congruent to the bisected interior angle $\measuredangle XCB$. The triangle ΔCBD therefore has two congruent base angles and is isosceles $|BC| = |CD|$. According to similarity of the triangles, C divides the segment AD in the same ratio as X divides segment AB. This shows that X is the inner point of division to the ratio $\frac{b}{a}$.

In the case of the outer point of division Y, the proof proceeds analogously; here, a parallel to CY is to be drawn through B; the point of intersection with AC is D (Fig. 15.10).

Because angle $\measuredangle DBC$ is congruent to angle $\measuredangle BDC$ the ΔDBC is isosceles again. Because of the similarity D divides the line segment AC in the same ratio as B

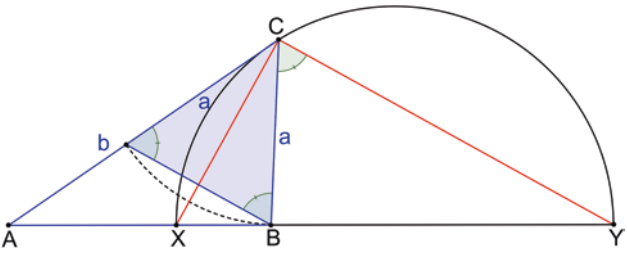

Fig. 15.10 Ratio of segments in an external bisector

divides the line segment AY. This shows that Y is the outer point of division. Since the points A, B have the distance ratio $\frac{b}{a}$ to three points C, X, Y, they lie on the Apollonius circle to AB. Since the angle bisectors of an internal or external angle are perpendicular to each other $XC \perp CY$„ the Apollonius circle to AB is also the Thales circle over XY. ($AXBY$) are therefore harmonic points.

15.3 The Touching Problem of Apollonius

The touching problem of Apollonius was handed down by Pappos and consists ofthe following universal exercise: Construct a circle which 1. passes through three (different) given points 2. touches a straight line and passes through two points 3. touches a circle and passes through two points 4. touches two straight lines and goes through one point 5. touches a straight line and a circle and goes through one point 6. touches two circles and passes one point 7. touches three straight lines 8. touches two straight lines and one circle 9. touches two circles and one straight line 10. touches three circles

Apollonius wrote two books *On Contact* about the problem, which are now lost. Some sub-problems are easy to solve, like (1), where the circle of the 3 vertices is the solution, or (7), where the incircle touches the three sides from the inside or the three excircles touch from the outside (4 solutions). Other sub-problems are considerably more difficult to solve, because either a large number of case distinctions are necessary or higher geometry tools are required. Therefore, this task served for centuries as a playground for the most famous mathematicians

- Francois Viète (Vieta) wrote the book *Apollonius Gallus* about it
- Rene Descartes used the problem in letters to Princess Elizabeth of the Palatinate
- Isaac Newton used some problems in his book *Principia*
- G.F. Marquis de l'Hôpital treated the task in his treatise on conic sections
- Thomas Simson used the task in his *Treatise of Algebra*
- Leonhard Euler published about it in 1788 at the Academy in Petersburg

Figure 15.11 shows 8 solutions to (10).

Another Apollonius Circle
The case where a circle is to touch three others is also solved by the (additional) Apollonius circle, which is inside the touching circles of a triangle (Fig. 15.12). Interestingly, there is also a transversal intersection point here. If you connect the touching points of the touching circles with the Apollonius circle with the opposite corner point of the triangle, the connecting lines intersect in a point called the Apollonius point.

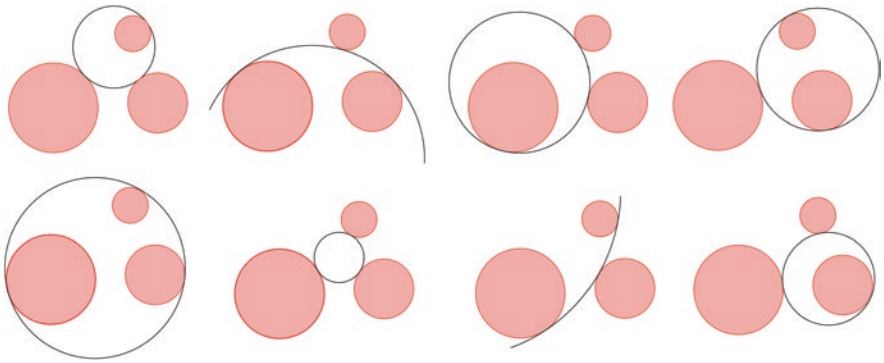

Fig. 15.11 For the touching problem

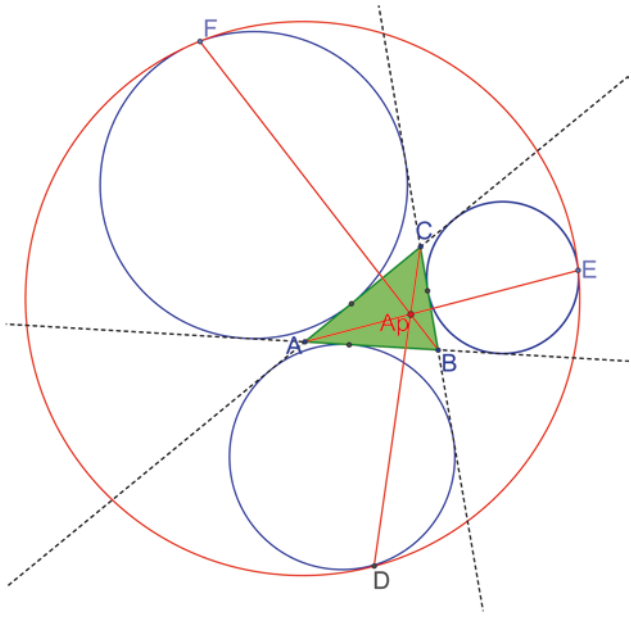

Fig. 15.12 Special Apollonius circle

15.4 The Theorem of Apollonius

This sentence is from the lost work *De locis planis* by Apollonius and is preserved by Pappus in proposition (VII, 122) of his *Collectio.*

If m is the median DC in the general triangle ΔABC, then (cf. Fig. 15.13a)

$$a^2 + b^2 = 2\left(m^2 + p^2\right)$$

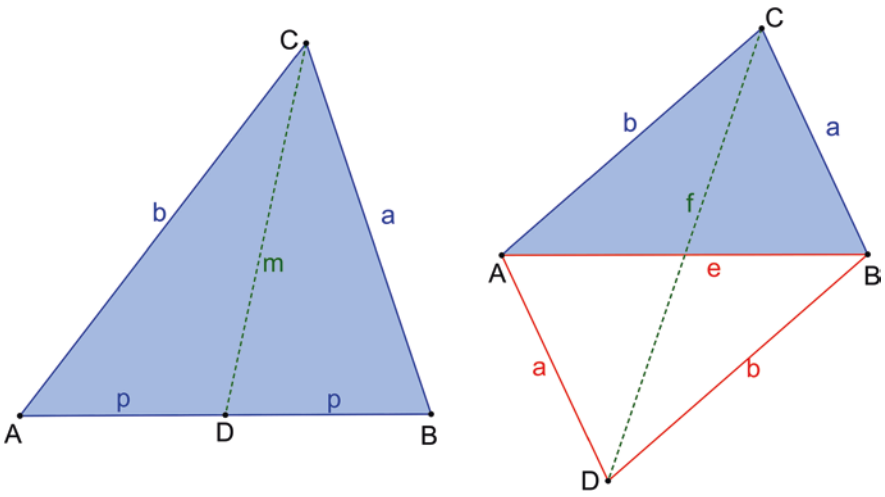

Fig. 15.13 Apollonius and Parallelogram Theorems

By point reflection, a parallelogram is obtained. Since the diagonals bisect each other in the parallelogram, it is equivalent to the *Parallelogram-Theorem:*

If a parallelogram has sides a, b and diagonals e, f, then (Fig. 15.13b):

$$2\left(a^2+b^2\right)=e^2+f^2$$

The theorem was found by Pappus (Collection VII, 122). A generalization is the following theorem:

Stewart's Theorem

The statement named after M. Stewart (1746) reads (Fig. 15.14):

$$a^2p+b^2q=c\left(d^2+pq\right)$$

Proof: Application of the cosine theorem to the partial triangles ADC and CDB yields:

$$b^2=d^2+p^2-2pd\cos\delta$$

$$a^2=d^2+q^2-2qd\cos\left(\pi-\delta\right)$$

$$=d^2+q^2+2qd\cos\delta$$

Multiplication of (2) with q or of (1) with p after addition shows

$$a^2p+b^2q=d^2p+q^2p+2pqd\cos\delta+d^2q+p^2q-2pqd\cos\delta$$

$$\Rightarrow a^2p+b^2q=d^2\left(\underbrace{p+q}_{c}\right)+pq\left(\underbrace{p+q}_{c}\right)=c\left(d^2+pq\right)$$

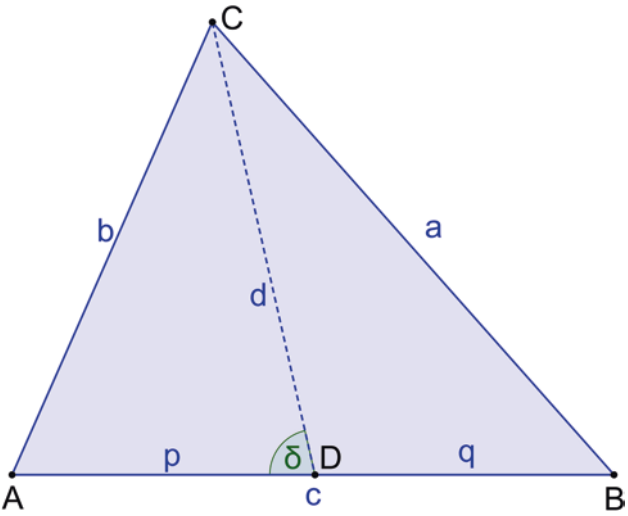

Fig. 15.14 To Stewart's statement

For the median one obtains with $p = q$ or $c = 2p$ again the theorem of Apollonius as a special case:

$$a^2p + b^2p = 2d^2p + 2p^3 \Rightarrow a^2 + b^2 = 2\left(d^2 + p^2\right)$$

What we know of Apollonius's achievements, whether from his own original texts or from the testimony of later mathematicians, nevertheless justifies the conclusion that the Hellenistic age produced in his person a mathematician of the first rank, in whose works Greek geometry reached its highest development.

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Beginnings of Trigonometry

16

The beginnings of trigonometry are found in the astronomy of the Greeks. The most important contributions to its beginning were made by *Aristarchos, Hipparchos* and *Menelaos*. The results of the latter two are taken over by K. *Ptolemaios* in his *Almagest*, whereby he omits the share of Menelaos.

Almagest I, 10: An inequality important for the following sections is in modern notation

$$\frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta};\quad \alpha < \beta < \frac{\pi}{2}$$

The left side of the inequality reads with Ptolemaios: If two unequal chords are drawn in a circle, the larger has to the smaller a ratio that is smaller than the circumference of the larger sector to the smaller.

If the chords $|AB| < |BC|$ are given, then it is to be shown:

$$\text{chord}(AB) : \text{chord}(BC) < arc(AB) : arc(BC)$$

Ptolemy's proof: In the circumcircle of the $\triangle ABC$ the bisector of the angle $\measuredangle ABC$ is drawn; the intersection of the bisector with AC is E resp. with the circle is D. From D the perpendicular is dropped upon AC; the foot point is F. The circle arc with the center D through E intersects the line AD in G and the line DF in H (Fig. 16.1).

It holds $|AD| = |DC|$, since both chords have congruent angles at the circumference. The angles at B are congruent because of the angle halving [Euclid III, 29]; DF is therefore the perpendicular bisector of the isosceles triangle $\triangle ADC$. Furthermore, according to Euclid [VI, 3] $|AE| < |EC|$. In addition, the following can be seen:

$$\triangle ADE > sector(DEG) \;\therefore\; \triangle DEF < sector(DEH)$$

It follows that

$$\triangle DEF : \triangle ADE < sector(DEH) : sector(DEG)$$

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_16

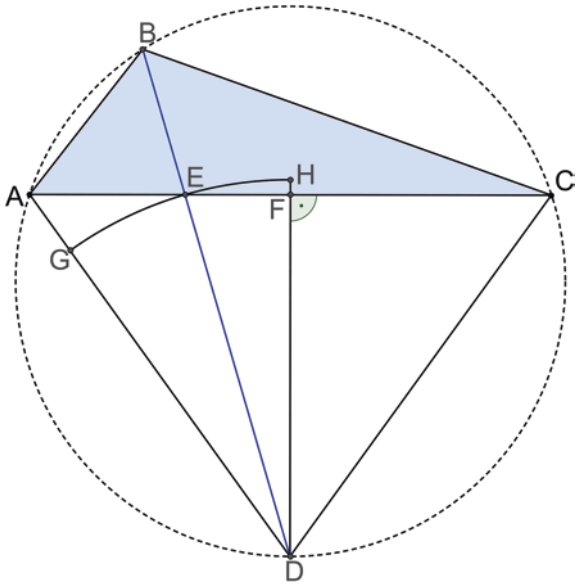

Fig. 16.1 Figure according to Ptolemy

According to Euclid [VI, 33], angles in equal circles behave like the corresponding arcs. Similarly, according to Euclid [VI, 1] applies: If the altitudes are equal, the triangle areas behave like the base lines. This provides

$$|FE| : |EA| < \measuredangle FDE : \measuredangle EDA \ \therefore |FA| : |EA| < \measuredangle FDA : \measuredangle EDA$$

Analogously, it follows that

$$|AC| : |EA| < \measuredangle CDA : \measuredangle EDA \therefore |CE| : |EA| < \measuredangle CDE : \measuredangle EDA$$

Because of the angle halving, it follows that $|CB| : |BA| = |AB| : |EC|$ and $\measuredangle CDE : \measuredangle EDA =$ Arc (CB) : Arc (BA). Because of the common chord BC or CD, the following angles are congruent: $\sphericalangle CDE = \sphericalangle CAB$ and $\sphericalangle ADE = \sphericalangle ACB$. In total, it follows that

$$\frac{|CB|}{|BA|} < \frac{\measuredangle CAB}{\measuredangle ACB} \Rightarrow \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta}$$

This is the claim.

To prove the right inequality, we use a figure from Archimedes' *Sand Reckoner* (Fig. 16.2) with the comment of F. Commandino. The figure can be found in a similar form in Euclid's *Optica,* 8.

It holds $|BD| = |FE|$ and $|CD| = |AF|$. The intersection of EF with AB is G. Because of $|AE| > |AG| > |AF|$, the circle arc around A at point G, the line segment AE at H or AD at K is cut.

Here it holds $\tan\alpha = \frac{|BD|}{|CD|}$ and $\tan\beta = \frac{|BD|}{|AD|}$. So it has to be shown that

$$\frac{|AD|}{|CD|} > \frac{\alpha}{\beta}$$

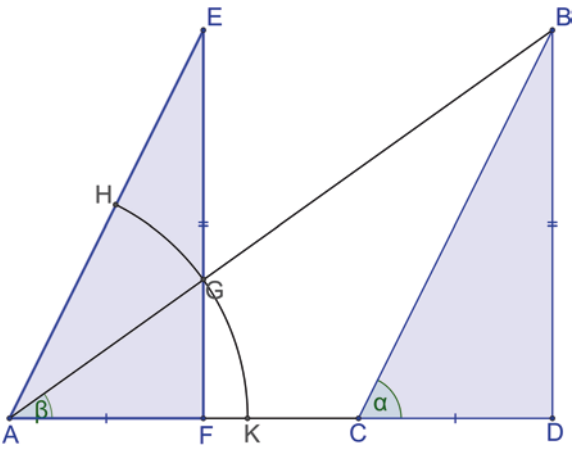

Fig. 16.2 Figure after Archimedes/Commandino

Because of the congruence of the triangles $\triangle AFE$ or $\triangle CDB$ it also holds $\measuredangle EAF = \alpha$. It holds

$$\measuredangle EAG : \measuredangle GAF = Sektor(HAG) : Sektor(GAK) < \triangle EAG : \triangle GAF = |EG| : |GF|$$

Similarly, we have

$$\measuredangle EAF : \measuredangle GAF < |EF| : |GF|$$

The latter transformation uses the fact that, at the same height, the areas behave like the sides. Because of $|EF| : |GF| = |BD| : |GF| = |AD| : |AF| = |AD| : |CD|$ it follows that

$$\frac{\measuredangle EAF}{\measuredangle GAF} < \frac{|AD|}{|CD|} \Rightarrow \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$

16.1 Aristarchus of Samos

Aristarchus (Αριστταρχος) lived around 310 to 230 BC. He was born on Samos and is said to have been a student of Aristotle's successor *Straton* of Lampsacus. In addition to his stay in Athens, he is also said to have lived in Alexandria. The only work of his that has been preserved *On the Sizes and Distances of the Sun and Moon*[1] is independent of the geo- or heliocentric world view. Archimedes reports in his treatise *On the Measurement of the Circle,* that Aristarchus placed the sun at the center:

[1] Heath T.L.: Greek Astronomy, Dover Publications, Reprint 1991, pp. 100–108.

> You, King Gelon, know that the 'universe' the astronomers call that sphere of which the earth is in the center, and the radius of which is equal to the distance between the center of the sun and the center of the earth. This is the general opinion, as you have heard it from astronomers. But Aristarchus has written a book consisting of certain hypotheses, and based on these assumptions, he shows that the universe is much larger than the 'universe' I just mentioned. His hypotheses are that the fixed stars and the sun are motionless, that the earth moves around the sun in the circumference of a circle, with the sun in the middle of this orbit, and that the sphere of the fixed stars, of which this sun is the center and within which the earth moves, has such a great extension that the distance from the earth to this sphere is equal to the distance of this sphere to its center, that the astronomers call that sphere the universe in which the earth is the center … This is the general opinion, as you have heard it from astronomers.

His writing presupposes several hypotheses, namely,

3. that, if the moon appears to us to be halved, the great circle of the moon, which separates the illuminated half from the dark one, falls in the line of sight (to the earth),
4. that, if the moon appears to us to be halved, its distance angle is less than a quadrant, reduced by a thirty-first $\left[R - \frac{1}{31}R = \frac{30}{31}R = 87{,}1^\circ\right]$,
6. that the moon takes up a fifteenth of a zodiacal sign $\left[\frac{1}{15}30^\circ = 2^\circ\right]$. (This statement was later revised by Aristarchus to $\left(\frac{1}{2}\right)^\circ$ as Archimedes reports.)

Aristarchus recognized that to the phase of the half-moon the moon C encloses a right angle with the line of sight and the sun. He postulated that the angle $\measuredangle ABC$ in the triangle sun (A), earth (B) and moon (C) has the value 87° mentioned in hypothesis (4). A measurement of such an angle is strongly questioned by O. Neugebauer[2]; he means that Aristarchus only makes a theoretical suggestion here. With this value Aristarchus was able to determine the moon distance in multiples of the Earth radius. With knowledge of the sine function, he would have received the result directly

$$\cos 87^\circ = \frac{|BC|}{|AB|} = \sin 3^\circ \Rightarrow \frac{|AB|}{|BC|} = \frac{|BC|}{\sin 3^\circ} = 19{,}1$$

He therefore had to determine the distance elementarily. The square $ABED$ is drawn over $|AB|$ as an auxiliary figure. The extension of BC results in the intersection point H with the square. Furthermore, BG is the bisector of the angle $\measuredangle FBE$. The intersection of the quarter circle ABE with the parallel $HK \parallel BE$ is D. On the circumference of the triangle BDK the radius $r = |BL|$ is applied downwards.

The following inequality derived above is used (Fig. 16.3).

Step 1) According to the assumption, $\measuredangle ABC = 87^\circ$ or $\measuredangle HBE = \alpha = 3^\circ$ and $\measuredangle GBE = 22.5^\circ$ Therefore, it follows that

[2] Neugebauer O.E.: Astronomy and History I, New York 1983, p. 381.

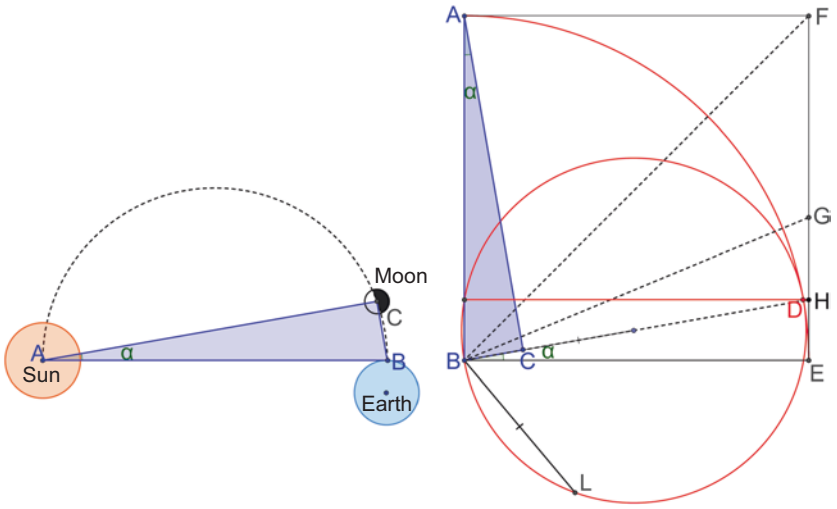

Fig. 16.3 Figure for the relative distance to the sun according to Aristarchus

$$\frac{|GE|}{|HE|} = \frac{\tan \measuredangle GBE}{\tan \measuredangle HBE} > \frac{\measuredangle GBE}{\measuredangle HBE} = \frac{22{,}5^\circ}{3^\circ} = \frac{15}{2} (*)$$

Since the bisector in $\triangle\, FBE$ divides the opposite side in the ratio of the adjacent sides, it holds

$$\frac{|FG|}{|GE|} = \frac{|FB|}{|BE|} \Rightarrow \left(\frac{|FG|}{|GE|}\right)^2 = \left(\frac{|FB|}{|BE|}\right)^2 = \frac{2}{1}$$

With the approximation already known to Plato for $\sqrt{2} = \frac{7}{5}$ we find that

$$\frac{|FG|}{|GE|} = \frac{\sqrt{2}}{1} > \frac{7}{5} \Rightarrow \frac{|FE|}{|GE|} = \frac{|FG| + |GE|}{|GE|} = \frac{|FG|}{|GE|} + 1 > \frac{12}{5}$$

Together with (*) it follows that

$$\frac{|AB|}{|BC|} = \frac{|FE|}{|EH|} = \frac{|FE|}{|GE|}\frac{|GE|}{|EH|} > \frac{12}{5} \cdot \frac{15}{2} = 18$$

Step 2) It is true that $\measuredangle BDK = \measuredangle DBE = 3^\circ = \frac{1}{30}R$, so that the arc $\widehat{BK}\,\frac{1}{60}$ of the full circle is. If $r = |BL|$ is the radius of the circle, then the arcs BK and BL behave like

$$\frac{\widehat{BK}}{\widehat{BL}} = \frac{\frac{1}{60}}{\frac{1}{6}} = \frac{1}{10} \Rightarrow \frac{\widehat{BK}}{\widehat{BL}} < \frac{|BK|}{r}$$

In the last transformation, the monotonicity was used; this provides

$$r < 10\,|BK| \;\therefore\; |BD| < 20\,|BK|$$

Because of the similarity of the $\triangle ABC$ or $\triangle BDK$ it follows that

$$\frac{|AB|}{|BC|} = \frac{|BD|}{|BK|} < 20$$

In total, the desired inequality results in

$$18 < \frac{|AB|}{|BC|} < 20 \Leftrightarrow 18 < \frac{1}{\sin 3^\circ} < 20$$

The ratio of the distances of the sun and the moon is therefore a number between 18 and 20. According to T. Heath (1991, p. 103) Aristarchus has also given the following inequalities for ratios of diameters D (Theorem 15, 17):

$$\frac{19}{3} < \frac{D(sun)}{D(earth)} < \frac{43}{6} \therefore \frac{108}{43} < \frac{D(earth)}{D(moon)} < \frac{60}{19}$$

16.2 Hipparchus of Nicaea

Hipparchus of Nicaea (today Isnik / Turkey) lived approximately from 190 to ca. 120 BC; according to Pappus he spent the greater part of his life on Rhodes. In literature the incorrect reference is found that he also lived in Alexandria, since he found a precise value of the local latitude . Hipparchus is considered the founder of scientific astronomy; as the author of the first table of chords he contributed to the development of trigonometry. His life data can be deduced from the careful observations which he made between 147 and 127 BC, which were taken over in Ptolemy's *Almagest*. O. Neugebauer writes in his three-volume history of astronomy:

> Even the most cursory report on ancient astronomy will not be able to avoid calling Hipparchus the *greatest astronomer* of antiquity. It is clear that the classification of sizes is not a precise description; nevertheless, this predicate will be a standing expression in the history of the natural sciences.

The only writings of Hipparchus that have survived are a commentary on a poem by Aratus on astronomy (third century BC) and some *geography-fragments.* The best source is the *Almagest,* since Claudius Ptolemy regarded him as his most important predecessor and integrated essential parts of his work (approximately 800 star names, the idea of a table of chords) into his *Almagest.* Using the results of Hipparchus, he was able to determine the relative distance to the moon by means of a diagram in Fig. 16.4, which can already be found in Aristarchus.

The angles $\measuredangle ZSE = \alpha$ or $\measuredangle ZM_1E = \beta$ are called solar (☉) or lunar (☾) parallax. According to Aristarchus, $\beta = 19\alpha$. For the triangles $\triangle SZM_1$ and $\triangle EZM_1$, the outer angle theorem [Euclid I, 32] applies:

$$\alpha + \beta = r + \rho$$

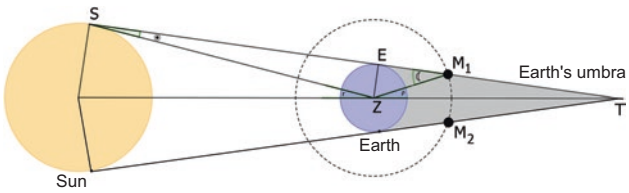

Fig. 16.4 Figure for the relative distance to the moon (not to scale)

The angle under which the solar radius is seen from Earth was known as $r = 15{,}5'$; the angle ρ is according to Ptolemy (IV, 8) the 2 ½ -fold of r. The desired lunar parallax results in

$$\beta + \frac{1}{19}\beta = 15{,}5'(1 + 2{,}5) \Rightarrow \beta = 51{,}5'$$

In $\triangle\, EZM_1$, it applies:

$$\sin\beta = \frac{|EZ|}{|ZM_1|} \Rightarrow |ZM_1| = \frac{|EZ|}{\sin\beta} = \frac{R_\oplus}{\sin 51{,}5'} = 66.8\, R_\oplus$$

The lunar distance is 66.8 Earth radii according to Hipparchus.

In addition to this measurement, Hipparchus also carried out further experiments to determine the distance to the moon. One of these measurements, which is based on a simultaneous observation of a solar eclipse on the Hellespont and from Alexandria, could be reconstructed by G.J. Toomer[3]. Another possibility is the following:

The size of the angle $2\rho = \measuredangle M_1 Z M_2$ can be estimated using the moon's movement. It was known that the moon moves relative to the sun and thus also to the earth's shadow by 12.2° per day, i.e. during the average duration of the lunar eclipse by $2\,{}^2/_3\,h$ the angle $2\rho = 81.3$ '. With the value $\rho = 40.7$ ', the lunar parallax and relative lunar distance are:

$$\beta = 54.1' \;\therefore\; |ZM_1| = 63.5\, R_\oplus$$

16.3 Menelaus' Theorem

Menelaos of Alexandria (70–140 AD) probably moved to Rome soon after his training in Alexandria. For two of his astronomical observations, which he made in Rome in 98 AD, have been preserved in Ptolemy's *Almagest* [VII, 3]. The place of birth Alexandria is reported by Pappus (*Collectio* VI) and Proclus (Euclid

[3] Toomer G.J.: Hipparchus on the distances of the sun and moon, Archive for History of Exact Sciences 14 (1974), pp. 126–142.

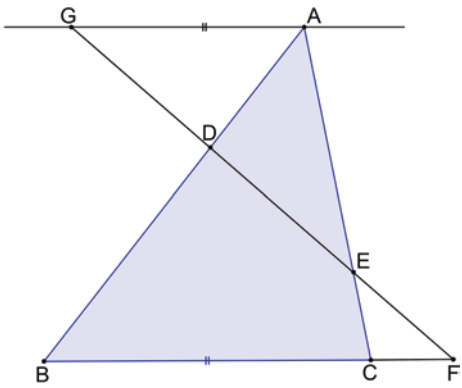

Fig. 16.5 Proof of the Theorem of Menelaus

commentary). Also the Arabic source *al-Fihrist* (Index) by ibn al-Nadim mentions him:

> He lived before Ptolemy, as the latter reports about him. He wrote a book on spherical geometry … The three books on the elements have been translated by Thabit ibn Qurra. Some of his books 'On Triangles' have been translated into Arabic.

Of his works, only one book *Sphaerica* is preserved in Arabic translation. In this book on spherical geometry, the theorem named after him, Menelaos (here for plane geometry) can be found:

> If a transversal intersects the sides of a triangle (or their extension) in three points, then the product of the corresponding ratios is 1.

Given the triangle ABC; the transversal FG intersects the sides of the triangle (or their extension) at the points D, E, F (Fig. 16.5). So it has to be shown that

$$\frac{|AD|}{|DB|} \cdot \frac{|BF|}{|CF|} \cdot \frac{|CE|}{|EA|} = 1$$

Some authors use directed lines here. Since the lines $\overrightarrow{AD}$ and $\overrightarrow{DB}$ are directed oppositely, a negative sign is added.

Proof

The parallel to BC through A is used as a auxiliary line; the intersection of the transversal with the parallel is G. The triangles $\triangle AGD$ and $\triangle BDF$ are similar, since they correspond in two pairs of angles: the opposite angles at D and the alternate angles $\measuredangle AGD$ and $\measuredangle CFE$ (at parallels). So the proportion applies

$$\frac{|AD|}{|BD|} = \frac{|AG|}{|BF|} \Rightarrow |AG| = \frac{|AD|}{|BD|}|BF|$$

Analogously, the triangles $\triangle AGE$ and $\triangle ECF$ are similar because they match in the opposite angles at E and the alternate angles $\measuredangle DAG$ resp. $\measuredangle ECF$ (at parallels). It follows

$$\frac{|AE|}{|EC|} = \frac{|AG|}{|CF|} \Rightarrow |AG| = \frac{|AE|}{|EC|}|CF|$$

Finally, setting them equal results in

$$\frac{|AD|}{|BD|}|BF| = \frac{|AE|}{|EC|}|CF| \Rightarrow \frac{|AB|}{|DB|} \cdot \frac{|BE|}{|EC|} \cdot \frac{|CF|}{|FA|} = 1$$

The Menelaos theorem is reversible; with the converse, one can check whether three points are collinear.

Converse: If for three points D, E, F of the triangle sides $\triangle ABC$ or their extensions

$$\frac{|AB|}{|BD|} \cdot \frac{|BE|}{|EC|} \cdot \frac{|CF|}{|FA|} = 1,$$

then they lie on a line.

Contradiction proof: Assume that the line DE does not intersect the triangle side AC at point F, but at point $G \neq F$. Then, according to Menelaos

$$\frac{|AB|}{|BD|} \cdot \frac{|BE|}{|EC|} \cdot \frac{|CG|}{|GA|} = 1$$

With the assumption, it follows

$$\frac{|CF|}{|FA|} = \frac{|CG|}{|GA|}$$

This is a contradiction because two different points cannot have the same ratio in a certain interval.

16.4 Applications in Geography

A important task of geography was the location determination. In order to obtain secure values for the coordinates, not only instruments but also calculation methods were necessary. A simple device was the gnomon (shadow indicator) in the form of a vertical rod on a plane or inside a hollow hemisphere. According to Diogenes Laertios (II, 1), Anaximander (610–546 BC) invented the gnomon, which he erected as a sundial in Sparta (around 570 BC).

A) Determination of Latitude

A simple determination of the latitude φ can take place at the equinox. For the sun's height at the equinox *(Aequinoctium)* $h = 90° - \varphi$; the latitude is the

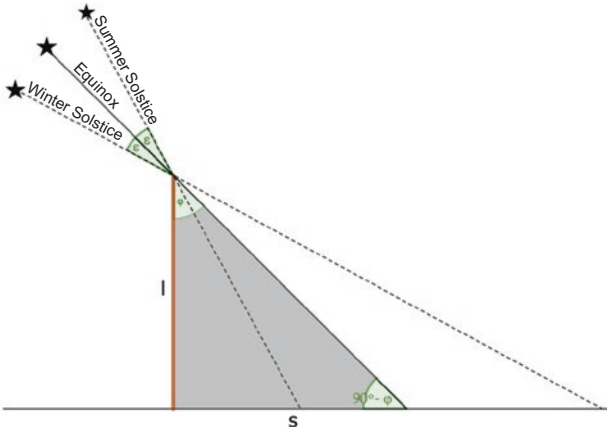

Fig. 16.6 Sun position angle at the gnomon

complementary angle φ can be determined using a gnomon (see Fig. 16.6). If s is the shadow length and ℓ is the gnomon length, then:

$$\tan \varphi = \frac{s}{\ell}$$

Of course, the Greeks were unaware of the tangent function, but they still knew the meaning of the quotient. Vitruv (Arch. IX, 7, 1) provides the following table of latitudes:

City	Ratio s/ℓ	Latitude φ
Rome	8/9	41°40′N
Athens	3/4	36°52′N
Rhodes	5/7	35°35′N
Tarent	9/11	39°20′N
Alexandria	3/5	31°00′N

Since Vitruvius could not make these measurements himself, it is assumed that he used older values from Aristarchus. The largest sundial (approx. 30 m high) was built on the *Field of Mars* in Rome by Augustus (next to his Peace Altar Ara Pacis); it is described in detail by Pliny the Elder (Nat. hist. 36, 72). The shadow lines were recessed into the floor plates; remnants of these bronze markings of the meridian were not found until 1980/1981.[4]

The obliquity ε of the Ekliptik can also be determined over the course of the seasons. Eratosthenes determined the value as $\frac{11}{83}$ of the semicircle

[4] Buchner E.: Die Sonnenuhr des Augustus, Mainz 1982.

($\varepsilon = 23°51'$), Ptolemy (I, 15) later the value $\varepsilon = 23°51'20''$ more accurately. The first quote shows that angles were always given as fractions of right angles or full circles before the second century BC, the 360° scale was only introduced by *Hypsikles,* the author of the book *Euclid* XIV, following the Babylonian model.

Another way to determine the latitude is by measuring the length of the day on the longest day (summer solstice). Pliny (II, 186) knows the dependence on latitude:

> It happens that, because of the different increase in the length of the day, the longest day in Meroë is 12 equinoctial hours and 8 parts of an hour, but in Alexandria it is 14, in Italy 15, and in Britannia 17.

The calculation of latitude from the longest day is based on spherical geometry and was explained by K. Ptolemy in the *Almagest* (II, 3). Later, in his *Geography* (I, 23), he provided ready-made tables for it.

Day length summer solstice	Latitudeaccording to Ptolemy
12 *h*	0°
12.5 *h*	8°55′N
13 *h*	16°25′N
15 *h*	40°55′N
18 *h*	58°N

Ptolemy probably took the values for higher latitudes from *Pytheas* from Massilia (Marseille), who (passing by the Pillars of Hercules) traveled along the northwest coast of Europe beyond Scotland and there heard of the mythical island of Thule. Whether he reached the Arctic Circle is not known; he reports, however, on the phenomenon of the midnight sun. His report evokes strong criticism from Strabo. The duration of the longest day can be represented graphically for different latitudes (Fig. 16.7):

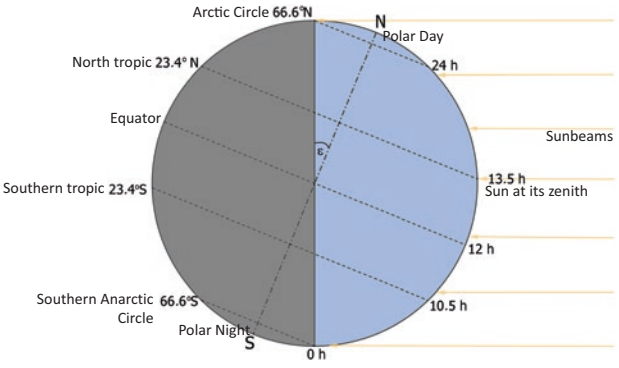

Fig. 16.7 Duration of the longest day for different latitudes

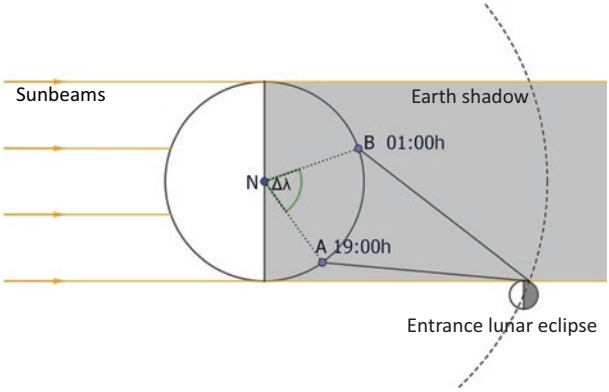

Fig. 16.8 Measurement of longitude

B) Determining Longitude

It is much more difficult to solve the problem of determining longitude. The simplest method is to measure the meridian passage of the sun at two locations with a precise clock. From the time difference $\triangle t$ of the clocks, the difference of the longitudes $\Delta\lambda$ can be determined; here, the difference $\triangle t = 1\ h$ corresponds exactly to $\Delta\lambda = 15^{\circ}$.

As Strabo reports, Hipparchus had the idea of using astronomical observations made at different locations to determine longitude. The idea was later taken up again by Ptolemy. In Geogr. (I, 4, 2) he writes about the famous eclipse of the moon during the Battle of Alexander of Arbela [20. Sep. 331 BC]:

> On the other hand, most distances … are quite inaccurately transmitted, … because one did not consider eclipses of the moon observed at different times at the same time worthy of recording, like the one that occurred in *Arbela* at 5 o'clock, but in *Carthage* at 2 o'clock; this would result in how many equatorial degrees to the east or west the places are apart from each other.

Figure 16.8 shows the entry of a lunar eclipse, which is visible at location A at 7:00 p.m., at location B at 1:00 a.m. local time. The difference in local times $\Delta t = 6\,h$ corresponds to the difference in longitudes $\Delta\lambda = 90°$.

The problem with this method is that eclipses are not always visible in all places and the time measurement is not carried out with the same accuracy. So A. Stückelberger[5] complains that Ptolemy gives a difference of 3 *h* for the above-mentioned lunar eclipse, whereas the local times differed *only* by 2 *h* 14 *min*. The accuracy of ancient timekeeping, which can be achieved with a sundial or water clock, is *not* comparable to the precision of a modern measurement. The phenomenon of *equation of time* also has to be considered with sundials; i.e. solar time

[5] Stückelberger A.: Bild und Wort, Philipp von Zabern 1994, p. 56.

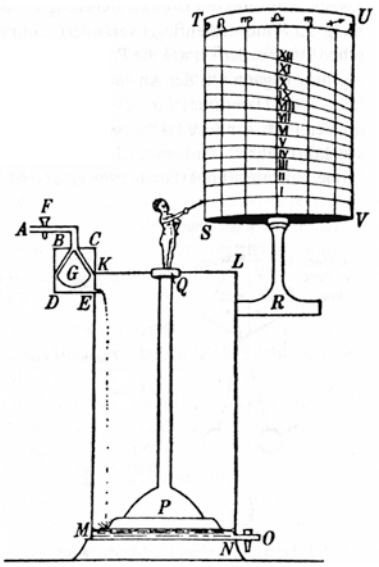

Fig. 16.9 Water clock of Ktesibios (Wikimedia Commons)

can deviate from local time by up to 20 min. Fig. 16.9 shows the water clock of *Ktesibios,* in which a swimmer indicates the hour depending on the season.

If the astronomical determination of length fails, the difference in longitude can only be determined by a distance measurement. Ptolemy calculates with 500 stadiums (= 105.7 km) for one degree at the equator.

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Heron of Alexandria

17

The life data of Heron of Alexandria (Ἥρων ὁ Ἀλεξανδρεύς) (Fig. 17.1) are not yet certain. According to the sources, he must have lived after Archimedes, but before *Pappos*, i.e. vaguely between 200 BC and 300 AD. A more precise survey of all lunar eclipses by O. Neugebauer[1] showed in 1938 that Heron probably lived in the first century AD. For in his work *Dioptra* a lunar eclipse is mentioned which was seen ten days before the vernal equinox. His statement that it occurred in Alexandria in the 5th (night) hour leads unambiguously to the lunar eclipse of 13 March 62 (Julian) for the time frame 200 BC to 300 AD.

The aforementioned dating of Neugebauer was by N. Sidoli[2] and others questioned. Neugebauer would have dealt too inaccurately with the information from Heron, so that other lunar eclipses could also be used for dating. Franz Krojer[3, 4], examined Sidoli's calculation and found it to be incorrect. Apparently, Sidoli determined the then vernal equinox using a NASA computer (!). Sidoli[5] now admits that Neugebauer's calculation *gives the best fit*, but has repeated his doubts about Heron's report. His main objection is that Heron states a difference of two hours in local time at the eclipse entry in Rome and Alexandria, but the true time difference (with modern clocks) is 1.1 h. The problem of ancient time measurement has already been mentioned in Sect. 16.4.

[1] Neugebauer O.: Über eine Methode zur Distanzbestimmung Alexandria-Rom bei Heron, Kongelige Danske Videnskabernes Selskabs Skrifter 26,2 (1938), S. 21–24.

[2] Sidoli N.: Heron of Alexandria's Date, Centaurus 53/1, 2011.

[3] Krojer F.: Astronomie der Spätantike, die Null und Aryabhata, Differenz-Verlag, München 2009.

[4] Krojer F.: Heronsgezänk. In: Astronomie der Spätantike, die Null und Aryabhata, München 2009, S. 31 ff.

[5] Sidoli N.: Heron of Alexandria's Date, Centaurus, 53 (2011), S. 55–61.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_17

Fig. 17.1 Picture of Heron from the translation of the Pneumatics into German (1688), (Wikimedia Commons)

That he lived at this time or earlier is confirmed by a quote from the Roman surveyor Columella, who wrote in 64 AD:

> … resulted in measurements for plane figures that match the formulas used by Heron, in particular for the equilateral triangle, the regular hexagon (not only the formula, but also Heron's sketch match here) and the circular segment, which is smaller than a semi-circle.

His lifetime can thus be approximately 10–85 AD.

It is unclear how to classify Pappos' report that *Heron the Mechanic* should be younger than Menelaos (ca. 70–140 AD). That Pappos (*Collectio* VIII) is the only author who names Alexandria as Heron's place of work is already a reason for H. Schellenberg[6] to question the assignment of location.

There are countless writings that function under the author name Heron. Many of his writings were used as lecture manuscripts for centuries, edited and supplemented with their own contributions. The book *Geometrica,* as E. Hoppe[7] (1918) has shown, comes from at least 2 manuscripts of different sources; J. Høyrup calls these manuscripts S or A,C. The work can therefore only be referred to as *pseudo-heronic*. The following writings are considered originals:

[6] H. M. Schellenberg: Notes on Heron of Alexandria and his work on the construction of artillery, A Roman Miscellany, Essays in Honour of A. R. Birley on his 70th Birthday, Ed. Schellenberg et al., Gdansk 2008, pp. 92–130.

[7] Hoppe E.: Is Heron the author of the definitions and geometry published under his name? Philogus 75 (1918), S. 202–226.

- De automatis (Automata)
- Dioptra (Devices and methods for distance measurement)
- Metrica (Geometry of figures and solids)
- Catroptica (Advanced optics)
- Pneumatica (Devices powered by pessure of air, steam and water)
- Belopoica (Machines of war)
- Mechanica (Simple machines)

The standard edition of the Heronian Books (Heronis Alexandrini opera quae supersunt omnia) appeared in 1899 at Teubner:

I) Pneumatica et automata
II) Heronis definitiones cum variis collectionibus. Heronis quae feruntur geometrica
III) Rationes dimetiendi et commentatio dioptrica
IV) Mechanica et catoptrica
V) Heronis quae feruntur stereometrica et de mensuris

The main work *Metrica* seemed to be lost for centuries, but the manuscript in Greek was discovered by J.L. Heiberg in Constantinople in 1896. The work consists of 3 books. Book I contains the calculation of triangles, quadrilaterals and so on and the surfaces of solids of rotational symmetry, Book II correspondingly the volumes of simple solids, Book III special cases of the division of areas and volumes, but also the determination of cubic roots. In Book III he writes himself that he *wants to collect mathematical knowledge.* In the literature, the *Metrica* edition by E.M. Bruins is hardly mentioned, which is part of the *Codex Constantinopolitanus* gr. 24 (1964) edited by him.

Generally, he is therefore considered as a *secondary* mathematician and imitator of automaton construction. H. Diels simply calls him a *ignoramus,* which is certainly too harsh a judgment. T. Heath[8] writes:

> The practical usefulness of Heron's lectures was so great that they were popular everywhere and the most popular of them were reprinted, revised and supplemented by later authors. This was inevitable with books like Euclid's Elements, which were used for instruction in Greece, Byzantium, Rome and Arabia for centuries.

B. van der Waerden does not appreciate him:

> Let us be glad that we have the masterpieces of an Archimedes and an Apollonius, and do not mourn the numerous lost arithmetic books of the Heron type.

[8] Heath, T.L.: A History of Greek Mathematics I, II, Oxford University Press 1931, Reprint Dover 1993.

Ingeborg Hammer-Jensen[9] sees him as ignorant, because in her opinion he copied large parts of the Pneumatica without understanding it. A.G. Drachmann[10] judges:

> A man who can always present his topic in such a way that everyone can easily understand it, is a man who *himself* understands it, and who is certainly not a fool or bungler.

M.S. Mahoney[11] notes:

> In the light of recent research, he [Heron] appears as a widely educated and often brilliant mathematician, as well as the living link to a continuous tradition of applied mathematics, which leads from the Babylonians via the Arabs to the Renaissance in the West.

Jens Høyrup[12] sees Heron in a Babylonian tradition, but trusts him to prove the area formula for triangles. He quotes the task *Geometrica* 24,3, the pattern of which can be found on the clay tablet BM 13901 #23 (but with different numbers) [cf. Mathematik im Vorderen Orient, S. 257].

The sum of the area and circumference of a square is 896. The area and circumference are to find separately.

Solution by Heron: 4 units are added to the side , whose half is 2 [feet], squared 4. Added to 896 gives 900, the square root of which is 30, reduced by the 2 units gives the remainder 28. So the area is 784 and the circumference 112 [feet]. The sum is 896 [feet] as given.

A geometric interpretation according to Heiberg is shown in Fig. 17.2. The modern solution is:

$$x^2 + 4x = 896 \Rightarrow (x+2)^2 = 900 \Rightarrow x = 28$$

Note that Heron solves all problems without the algebraic formalism that is used today. The application of modern formula language is historically not correct.

J. Høyrup also finds this task in later authors and can thus establish a tradition:

- Abu Bakr: Liber mensurationen (around 800)
- translated by Gerard of Cremona (around 1150)
- Savasorda: Liber embadorum (twelfth century)
- Leonardo von Pisa: Pratica geometrie (1220)
- Piero della Francesca: Trattato d'abaco (around 1460)
- Luca Pacioli: Summa de arithmetica (1494)

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[9] Hammer-Jensen I.: Die Heronische Frage, hermes 63(1928), S. 34–47.

[10] Drachmann, A.G.: The Mechanical Technology of Greek and Roman Antiquity, Munksgaard Copenhagen 1963.

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[12] Høyrup J.: Hero, Pseudo-Hero, and Near eastern practical geometry, Roskilde universitetscentre, Reprint 1996, Nr. 5.

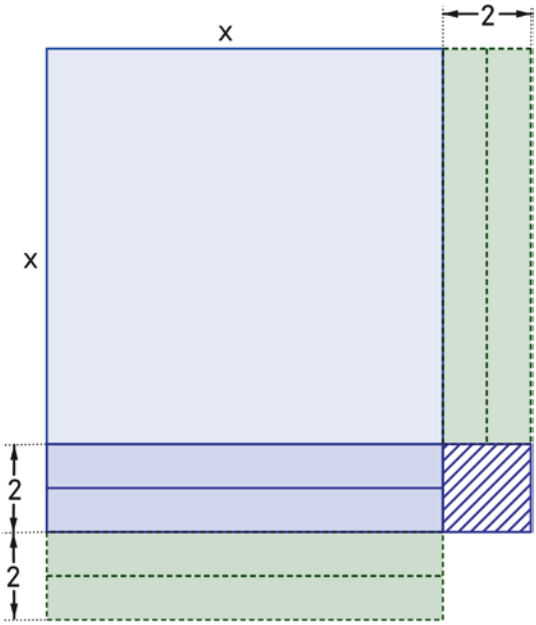

Fig. 17.2 Geometric interpretation of Geometrica 24,3

17.1 From the Definitions

The *Definitiones* have been considered pseudo-Heronian since Wilbur Knorr[13] (1993). They are of particular importance for the history of mathematics because it is possible that material from the *Definitiones* was subsequently interpolated into the Elements. That the definitions contained in the *Elements* are insufficient was already known in antiquity. Lucio Russo was able to show in (Forgotten Revolution, 2004) that definitions 1 to 7 of the Euclid's *Elements* were interpolated from *Definitiones* 1–2, 4, 8 and 9. Since W.R. Knorr[14] rejects Heron as the author, he attributes the *Definitiones* to Diophantus.

The Definitiones comprise (bilingual) the pages 33–198 of Volume III of the *opera omnia* by Heron. Therefore, only a few definitions can be printed here; the ones given here show clearly the difficulties of defining things so that the explanation is neither recursive nor relies on undefined facts.

1. [Point] A point is something that has no part or a boundary without extension or boundary of a line, and its essence is to be only conceivable by thought, because it is both without parts and without size. […] For from the

[13] W. R. Knorr: *Arithmetike stoicheiosis,* On Diophantes and Hero of Alexandria, Historia Mathematica 20 (1993), 180–192.

[14] W. R. Knorr, The wrong Text of Euclid, Science in Context, 14 (2001), 133–143.

movement of the point or rather from the idea of a point in flux, the idea of a line arises, and in this sense the point is the beginning of the line as the surface of a solid body.

2. [Line] But a line is a length without width and depth or what first assumes existence within magnitude, or what has extension in one dimension and is divisible, and it arises when a point slides from top to bottom by means of the concept of continuity, and is enclosed and bounded by points, while itself being the boundary of this surface. […]
4. [Straight Line] A straight line is one that has the same position with respect to the points on it, running parallel and stretched out between the endpoints. It is the smallest of the lines that have the same endpoints between two given points; it is such that all parts completely coincide with all parts. […]
8. [Area] An area is something that only has length and width, or the boundary of a body or space, or what has two-dimensional extension without depth, or the limiting surface of any solid and flat figure according to the two dimensions of length and width. It arises through the sliding of a line that slides from left to right in width. […]
27. [Circle] A circle is the plane enclosed by a line. The figure is therefore called a circle, the line enclosing it is called a circumference, and all the straight lines that reach this from *one* of the points inside the figure are equal to each other. […] But a circle is also called a line that forms equal distances in all parts. A circle arises when a straight line, while remaining in the same plane, is led around with one endpoint fixed until it is brought back into the same position from where it began to move.
36. [Lune] A lune is a figure enclosed by two arcs of a circle, one concave and one convex, or the difference between two circles that do not have the same center. […]
40. [Triangle] A triangle is a figure enclosed by three straight lines with three angles.
70. [Parallel] Parallels are called lines running at the same time, which are in the same plane and extended on both sides, after neither of them converge.
73. [Tiling of the plane] Of all plane, right-angled and equilateral figures but these alone fill the space of the plane: the triangle, the square and the hexagon. The triangle fills, when it takes five more from its apex, the space of the plane without leaving any space in between, and likewise the square, when it takes three, and the hexagon, when it takes two.
94. [Conic section] Cut through the tip, a cone produces a triangle as a cut, the base cut parallel a circle, not parallel but another line group, the conic sections called. […] Acute-angled is now the one that is connected, forming a shield-shaped figure, called by some ellipse. The cut of a right-angled cone is called a parabola, the obtuse-angled but hyperbola.
119. [Size] A size is what can be increased and divided into infinity; their kinds are lines, surfaces, solids. An infinite size but is one that a larger cannot be thought, whatever extension they have, so that it has no limit.

17.2 From Metrica and Geometrica

The numbering is from the author:

1. Given is the sum of the diameter, circumference and area of a circle. Find the quantities if the sum is 212 (Geom. 21.9).

Solution: If d is the diameter, then we get with the Archimedes approximation $\pi = \frac{22}{7}$

$$\frac{1}{4}\pi d^2 + \pi d + d = \frac{1}{4}\pi d^2 + (\pi + 1)d = \frac{11}{14}d^2 + \frac{29}{7}d = 212$$

Modern solution is: Simplify and expand with 11 shows after quadratic completion

$$121d^2 + 638d + 29^2 = 32648 + 29^2$$
$$(11d + 29)^2 = 183^2 \Rightarrow d = 14$$

Thus the circumference is $U = 44$, the area $A = 154$. This task can also be found in (*Geom.* 24, 27) with the given sum 67 ½. J. Høyrup claims that this problem also has Babylonian origins. Since he does not give a source, it cannot be confirmed.

2. A right-angled triangle is given by the sum of the area A and circumference, here 280. If s is half the circumference, then it should apply:

$$A + 2s = 280$$

If r is the inradius of the triangle, the following formulas can be assumed as known:

$$A = \frac{1}{2}ab = rs \ \therefore \ r + s = a + b$$

Thus the given sum can be factorized:

$$A + 2s = (r + 2)s = 280$$

Possible factors of the right side are $(2 \times 140), (4 \times 70), (5 \times 56), (7 \times 40), (8 \times 35), (10 \times 28), (14 \times 20)$. Heron knew that the area of a Heronian triangle has the divisor 6. We therefore choose the decomposition $r + 2 = 8 \Rightarrow r = 6$, since rs is also divisible by 6. At the same time $s = 35$ follows. According to Babylonian pattern

$$(b - a)^2 = (a + b)^2 - 4ab = (r + s)^2 - 8rs \Rightarrow b - a = \sqrt{(r + s)^2 - 8rs}$$

Insertion gives

$$b = \frac{b - a}{2} + \frac{a + b}{2} = \frac{1}{2}\left[\sqrt{(r + s)^2 - 8rs} + (r + s)\right] = 21$$
$$a = b - \sqrt{(r + s)^2 - 8rs} = 20$$

The sought triangle is here (20; 21; 29). Similar tasks can also be found with Diophantus, in the form that the sum of the area and one of the catheti is given or from the area plus the sum of the sides (catheti).

3. To find a Heronian triangle with an area of 5.

The calculation was also taken over by Diophantus. Since the area of a Heronian triangle is divisible by 6, the given area is extended with 36 here. To the content 180 one looks for a matching Heronian triangle; this is (9; 40; 41). Since the areas of similar triangles behave like the squares of the corresponding sides, the desired sides result from division by 6. The desired triangle is therefore $\left(\frac{3}{2}; \frac{20}{3}; \frac{41}{6}\right)$.

4. To find two rectangles (a, b) or (x, y) such that the circumference of the second is three times the first; the area of the first three times the second.

The general system for the multiple n is:

$$a + b = n(x + y) \ \therefore \ xy = nab$$

Heron does not give a calculation; according to H.G. Zeuthen one can proceed as follows. Since the system is underdeterminate, a multiple can be chosen freely, here with a new unknown z:

$$a = n \ \therefore \ b = nz$$

This makes it possible to write the system as

$$x + y = 1 + z \ \therefore \ xy = n^3 z$$

According to Babylonian pattern

$$(y - x)^2 = (x + y)^2 - 4xy = (1 + z)^2 - 4n^3 z = z^2 - \left(4n^3 - 2\right)z + 1$$

Since there is a square on the left, this also applies to the right side. If you simply set it equal to 1, you get

$$z = 4n^3 - 2 \ \therefore \ y - x = 1$$

This results in the general solution:

$$a = n \ \therefore \ b = n\left(4n^3 - 2\right) \ \therefore \ x = \frac{z}{2} = 2n^3 - 1 \ \therefore \ y = 2n^3$$

In particular, for $n = 3$ it follows:

$$(a, b) = (3; 318) \ \therefore \ (x, y) = (53; 54)$$

A similar task is as follows:

5. Two rectangles (a, b) or (x, y) of the same circumference are sought so that the area of the first is four times the area of the second.

The general system for the multiple n is:

$$a + b = x + y \ \therefore \ xy = nab$$

Again, Heron sets without comment

$$a = n - 1 \ \therefore \ b = n\left(n^2 - 1\right)$$

Inserting yields

$$x + y = (n - 1) + n\left(n^2 - 1\right) = n^3 - 1$$

This results in

$$(y - x)^2 = (x + y)^2 - 4xy = \left(n^3 - 1\right)^2 - 4n^2(n - 1)\left(n^2 - 1\right) = \left(n^3 - 2n^2 + 1\right)^2$$

Thus it follows:

$$y - x = n^3 - 2n^2 + 1 \ \therefore \ y + x = n^3 - 1$$

The sum and difference finally result in:

$$x = n^2 - 1 \ \therefore \ y = n^2(n - 1)$$

In particular, for $n = 4$ it follows:

$$(a, b) = (3; 60) \ \therefore \ (x, y) = (15; 48)$$

The task was later taken over by the Byzantine M. Planudes in his arithmetic book (cf. Mathematik im Mittelalter, S. 240).

6. Metrica I,8: Determine the area of the triangle with the sides 7, 8, 9, without calculating the altitude.

Solution: Using the Heron formula, he determines the area with $s = \frac{7+8+9}{2} = 12$:

$$\mathcal{F} = \sqrt{12(12 - 7)(12 - 8)(12 - 9)} = \sqrt{720}$$

Heron does not simplify $\sqrt{720} = 12\sqrt{5}$; but calculates the root in one step according to his method. The starting value is the integer $\sqrt{729} = 27$

$$\sqrt{720} \approx \frac{1}{2}\left(27 + \frac{720}{27}\right) = \frac{161}{6} = 26\frac{5}{6}$$

He writes the result as an Egyptian fraction $26\frac{1}{2}\frac{1}{3}$.

7. Metrica II, 12: Determine the volume of the spherical segment, given the base circle diameter $2x = 12$ and the height $h = 2$ (Fig. 17.3)

Solution: Heron quotes the sentence of Archimedes that the volume of a spherical segment is related to a cone (with the same radius and the same height) as the sum

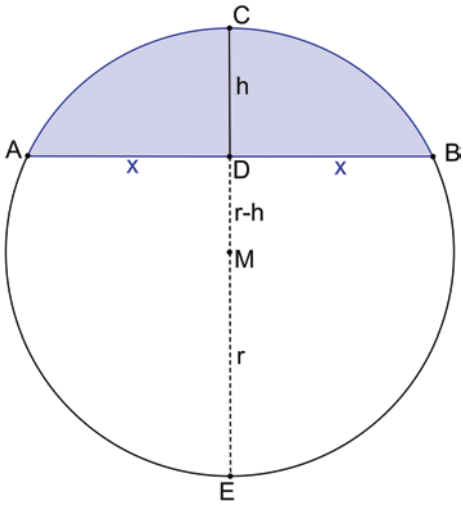

Fig. 17.3 Figure for spherical segment

of the sphere radius and the height of the remaining segment to the height of the remaining segment [Archimedes, Sphere and Cylinder, II, 2].

$$\frac{segment(ABC)}{cone(ABC)} = \frac{|EM| + |DE|}{|DE|}$$

According to the secant theorem [Euclid III, 35] it holds: $|AD||DB| = |CD||DE| \Rightarrow x^2 = h(2r - h)$. The sphere radius is therefore

$$r = |EM| = \frac{1}{2}\left(\frac{x^2}{h} + h\right) = \frac{1}{2}\left(\frac{36}{2} + 2\right) = 10 \Rightarrow |DE| = 2r - h = 18$$

Thus, the segment volume is the $\frac{10+18}{18} = \frac{14}{9}$-fold of the cone volume; this is $\frac{1}{3}\pi x^2 h = 24\pi$. With the approximation of Archimedes $\pi \approx \frac{22}{7}$ the desired segment volume results in

$$\frac{14}{9} \cdot 24 \cdot \frac{22}{7} = \frac{352}{3} = 117\frac{1}{3}$$

Heron knows the more accurate estimate for π by Archimedes. For he writes in Problem I, 26, that Archimedes in the (lost) work *Plinthides and Cylinders* determined the following bounds

$$\frac{211875}{67441} < \pi < \frac{197888}{62351}$$

8. Problem I, 34: The area of an ellipse with the large axis 16, the small 12 is sought.

Heron refers to Archimedes and writes that the product of the axes is equal to the square of the diameter of a circle that is equal in area to the ellipse. He calculates $12 \cdot 16 = 192$, from this he takes the share $\frac{11}{14} \approx \frac{\pi}{4}$; this gives him $146\frac{1}{2}$ instead of

the correct $150\frac{6}{7}$. The true value of the ellipse area is four times as much, since the specification obviously means semi-axes.

9A. Metrica III, 20: Given is a four-sided pyramid $ABCDE$ with the base $ABCD$, the apex E and the side length $|AE| = 5$. Through a plane parallel to the base plane, the pyramid volume is to be divided so that the cut-off pyramid apex is four times as large as the (remaining) pyramid stump (Fig. 17.4).

Solution: With (similar) pyramids, the volumes behave like the third powers of the corresponding sides.

$$\frac{pyramid(ABCDE)}{pyramid(FGHKE)} = \frac{5}{4} = \left(\frac{|AE|}{|FE|}\right)^3 \Rightarrow |FE|^3 = \frac{4}{5}|AE|^3 = 100$$

To determine $|FE| = \sqrt[3]{100}$. Because $4 < \sqrt[3]{100} < 5$ and the differences to the neighboring cubes $|100 - 64| = 36$ or $|125 - 100| = 25$ Heron interpolates in the following way

$$\sqrt[3]{100} \approx 4 + \frac{36 \cdot 5}{36 \cdot 5 + 25 \cdot 4} = 4\frac{9}{14}$$

The sought edge length $|FE|$ is $4\frac{9}{14}$. Heron knows the formula for the content of a capped 4-sided pyramid. If A_1, A_2 are the base and top areas, the formula applies

$$V = \frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$$

Heron writes this in the form

$$V = h\left(\frac{2}{3}\frac{A_1 + A_2}{2} + \frac{1}{3}\sqrt{A_1 A_2}\right)$$

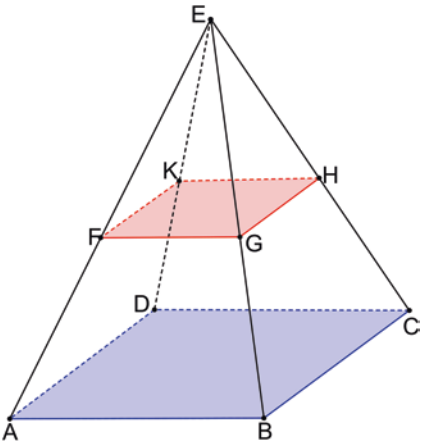

Fig. 17.4 Figure to the 4-sided pyramid

The bracket represents a weighted mean of the arithmetic or geometric mean.

9B. In addition to the above-mentioned proportion, the task (II, 8) contains a complicated term for the capped rectangular pyramid with a rectangular base and top; in modern notation it reads (Fig. 17.5):

$$V = \left[\frac{a+c}{2}\frac{b+d}{2} + \frac{1}{3}\frac{a-c}{2}\frac{b-d}{2}\right]h$$

The term can be simplified to

$$V = \frac{1}{6}(2ab + 2cd + ad + bc)h$$

This is a special case of the modern formula with base area A_1, top area A_3 and mean area A_2, which corresponds to Kepler's rule.

$$V = \frac{1}{6}(A_1 + 4A_2 + A_3)h$$

9C. Here is a task that Heron fails. He considers a frustum of a square pyramid whose base has the side $a = 28$ [foot], the top face the side $b = 4$ [foot] and the edge $k = 15$ [foot] is; the height h is sought

If you project the top onto the base, one corner of the top has the distance $d = \frac{1}{2}(28-4)\sqrt{2}$ to the corresponding corner of the base. If you look at the resulting inclined triangle, which forms an edge with the height b, you get:

$$h^2 = k^2 - d^2 \Rightarrow h^2 = 15^2 - \left(12\sqrt{2}\right)^2 < 0$$

But this leads to a contradiction! Heron continues to calculate with the amount here.

10. Task I, 35: The area of a parabola section is sought if its base is 12 and the height is 5.

The triangle inscribed in the parabola segment has the area $A = 30$. According to Archimedes, the parabola segment has the $\frac{4}{3}$–fold area, thus 40.

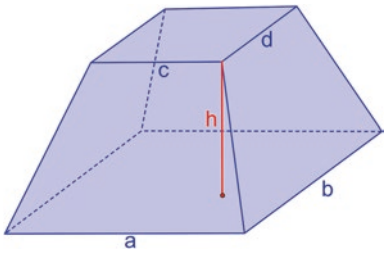

Fig. 17.5 Figure for the frustum of apyramid

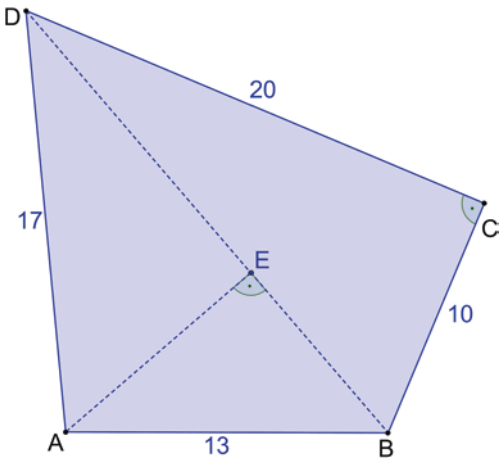

Fig. 17.6 Figure for the quadrilateral with a right angle

11. Task I, 14: The area of the quadrilateral $ABCD$ is to be found with $|AB| = 13, |BC| = 10, |CD| = 20, |AD| = 17$ and a right angle at C (Fig. 17.6).

Heron draws the diagonal BD and drops the perpendicular from A onto BD. According to Pythagoras, he gets $|BD|^2 = 500$. By means of altude theorem (Euclid II, 13), the square of the altitude in $\triangle ABD$

$$|AE|^2 = 96\frac{1}{2}\frac{1}{5}\frac{1}{10}$$

The fourfold area square of $\triangle ABD$ is thus to

$$4\mathcal{F}^2(ABD) = |AE|^2|BD|^2 = 96\frac{4}{5} \cdot 500 = 48.400$$

Taking the square root results in: $\mathcal{F}(ABD) = 110$. Since the area of the other part triangle $\mathcal{F}(BCD) = 100$ is, the desired surface area of the quadrilateral results in 210. The result $\mathcal{F}(ABD) = 110$ is exactly as one easily verifies by means of the cosine theorem:

$$\cos\alpha = -\frac{21}{221} \Rightarrow \sin\alpha = \frac{220}{221} \Rightarrow \mathcal{F}(ABD) = \frac{1}{2}|AB||AD|\sin\alpha = \frac{13 \cdot 17}{2}\frac{220}{221} = 110$$

17.3 From the Stereometrica

The writing *Stereometrica*[15] is, similar to the *Geometrica,* a work of Heron, which has subsequently undergone a processing for teaching purposes. The work is of great importance for the history of construction and architecture, as it served the

[15] Heronis Alexandrinis opera quae supersunt omnia, Band V, Teubner 1914.

training of architects at the court of Constantinople. An investigation of Heron's surveying and vaulting doctrine as the basis for the planning of the Hagia Sophia comes from Helge Svenshon[16]. Procopius of Caesarea has reported in his work *De aedificiis* (I, 1) (around 560) about the numerous buildings that were erected under Justinian's rule in the Roman Empire, especially in Ravenna.

Some typical problems are presented here from Book I of *Stereometrica*:

Problem I, 42
Calculate the seating capacity of a theater: Based on measurements, the top row of seats is 420 feet, the bottom row is 180 feet. There are 280 rows of seats.
Solution: The average number of seats is $\frac{1}{2}(420 + 180) = 300$. With 280 rows/steps, this makes a total length of seats to 84,000 feet. If you allow for one person per foot (about 30 cm), then the theater's capacity is 84 000 people.

Problem I, 43
Calculate the number of seats in the top row of a theater, if the bottom row holds 40 people and each higher row has 5 more people, for a total of 50 rows of seats.
Solution: Heron calculates the number of seats according to $(50 - 1) \cdot 5 + 40 = 285$, so correctly according to the formula for the arithmetic series.

Problem I, 44
An amphitheater has the length $\ell = 240$ feet, the width $b = 60$ feet. The area and circumference are to be found.
Solution: Heron calculates the area as follows: $A = \frac{11}{14} \cdot 240 \cdot 60 = 11314\frac{1}{4}\frac{1}{28}$. For the circumference he obtains $U = 2 \cdot 240 + 60 \cdot \frac{7}{6} = 550$ (feet). From the accompanying figure one recognizes that it is a rectangular area with two semi-circles attached at both ends. With the usual approximation for π used by Heron, the correct values are therefore $A = 180 \cdot 60 + \frac{11}{14} \cdot 60^2 = 13628\frac{4}{7}$ and $U = 2 \cdot 180 + \frac{22}{7} \cdot 60 = 548\frac{4}{7}$. The values obtained by Heron are therefore only an approximation. In a parallel problem he calculates the circumference according to the obviously empirical formula with the same result

$$U = 2\sqrt{\ell^2 + b^2 + \ell b} = 2\sqrt{240^2 + 60^2 + 240 \cdot 60} = 2\sqrt{75600} \approx 2 \cdot 275 = 550$$

Problem I, 50
A well has the (inner) diameter 5 feet, the wall thickness of the wall is 2 feet, the depth 20 feet. The interior volume of the well is to be found.
Solution: Heron calculates $\frac{11}{14} \cdot 20\left(9^2 - 5^2\right) = 880$, correctly as the difference of two cylinders.

[16] H. Svenshon: Das Bauwerk als *aistheton soma* – eine Neuinterpretation der Hagia Sophia im Spiegel antiker Vermessungslehre und angewandter Mathematik. In: Falko Daim, Jörg Drauschke (Ed.): Byzanz – Das Römerreich im Mittelalter, Monographien des RGZM, 84 (2,1). Mainz 2010, S. 59–95.

Task I, 51
A bucket has a lower diameter of 5 feet, the upper 3 feet; its height is 8 feet. The wine reaches a height of 6 feet. The volume of the contained wine is sought.
Solution: Heron first calculates the diameter of the bucket at the level of the liquid surface and obtains $\frac{7}{2}$ *feet,* the average diameter up to the filling height is thus $\frac{13}{4}$. According to the cylinder formula, the volume $\frac{11}{14} \cdot \left(\frac{17}{4}\right)^2 \cdot 6 = \frac{9537}{112} = 85\frac{17}{112}$ results, which Heron rounds to $85\frac{1}{7}$.

17.4 Heron's Formula

The proof of Heron's formula for the area of a triangle is found in his writings *Metrica*[17] (I, 8) and *Dioptra* 24. Al-Biruni writes in his book *About finding chords in a circle* that the formula is due to Archimedes. E.J. Dijsterhuis[18] states in his book on Archimedes (p. 412) that the proof is atypical for Archimedes, since products of areas, i.e. functions of the $\mathbb{R}^4$, occur here. The proof can therefore directly originate from Heron. The formula is

$$\mathcal{F}(\triangle) = \sqrt{s(s-a)(s-b)(s-c)};\ s = \frac{1}{2}(a+b+c)$$

Heron describes the formula with the following words:

> There exists a general method to determine the area of any triangle given the sides without the altitude. As an example, let's choose the triangle (7; 8; 9). Add 7 and 8 and 9, result is 24. Take the half, that makes 12. Subtract 7, difference is 5; subtract 8 again, difference is 4; subtract 9 once more, difference is 3. First, multiply 12 with 5, result is 60, then with 4, result is 240, finally with 3, result is 720. From that, take the square root, it provides the area of the triangle.

Proof by Heron:

Given is $\triangle ABC$ with the incircle (radius r) with center I and the points of contact D, E and F. Producing BC beyond B by the distance $|AF|$ yields the point H. Erecting perpendulars at the point I and at the point B intersect at the point G (Fig. 17.7). For the 3 partial triangles, which are formed by the incircle center, following areas result

$$|BC|r = 2\mathcal{F}(BCI)$$
$$|AC|r = 2\mathcal{F}(CAI)$$
$$|AB|r = 2\mathcal{F}(ABI)$$

The addition of the 3 equations yields with the circumference U

$$(|AB| + |AC| + |BC|)r = Ur = 2\mathcal{F}(ABC)$$

[17] Heronis Alexandrinis opera quae supersunt omnia, Band III, Teubner 1914.

[18] Dijsterhuis E.J.: Archimedes, Ejnaar Munksgaard 1956.

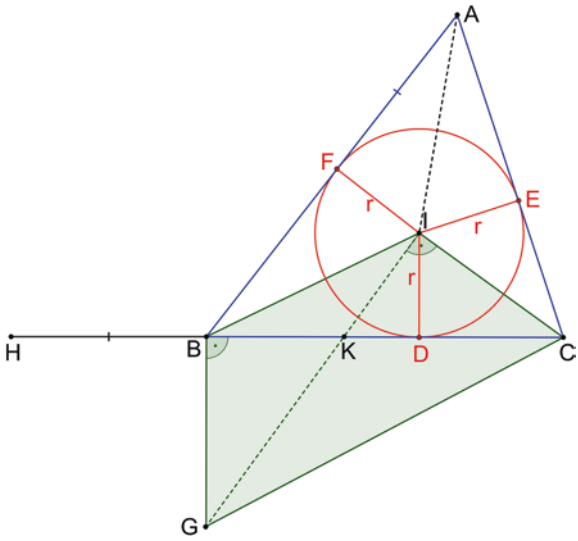

Fig. 17.7 Proof of the Heron's formula

Insertion of the tangent sections $|AE| = |AF| = |HB|, |CD| = |CE|$ and $|BF| = |BD|$ shows

$$\left(\underbrace{|AF|}_{|HB|} + \underbrace{|BF|}_{|BD|} + |BD| + |CD| + \underbrace{|CE|}_{|CD|} + \underbrace{|AE|}_{|HB|} \right) r = 2\mathcal{F}(ABC)$$

In summary, this yields

$$(|CD| + |BD| + |HB|)r = |CH|r = \frac{1}{2}Ur = \mathcal{F}(ABC) \quad (*)$$

Since the line CG appears from B and I at a right angle, B or I lie therefore on the Thales circle over CG. The points $BGCI$ thus form a cyclic quadrilateral. The opposite angles $\measuredangle BGC$ and $\measuredangle BIC$ in a cyclic quadrilateral add up 2 rights

$$\angle BIC + \angle BGC = 2R$$

Since the angle bisectors bisect the angles at the incircle center, the following angle sum results in half the full angle at I

$$\angle BID + \angle DIC + \angle AIF = \angle BIC + \angle AIF = 2R$$

In comparison, $\angle BGC = \angle AIF$. Therefore, the right-angled triangles $\triangle BGC$ or $\triangle AIF$ are similar. The following proportions therefore apply

$$\frac{|BC|}{|BG|} = \frac{|AF|}{|FI|} = \frac{|BH|}{|ID|} \quad \therefore \quad \frac{|BC|}{|HB|} = \frac{|BG|}{|ID|} = \frac{|BK|}{|KD|}$$

From the latter proportion, it follows by addition

$$\frac{|BC|}{|HB|}+\frac{|HB|}{|HB|}=\frac{|BK|}{|KD|}+\frac{|KD|}{|KD|}\Rightarrow\frac{|CH|}{|HB|}=\frac{|BD|}{|KD|}$$

Extending with $|CH|$ or $|CD|$ results in

$$\frac{|CH|^2}{|CH||HB|}=\frac{|BD||DC|}{|KD||CD|}=\frac{|BD||DC|}{r^2}$$

The last transformation $|KD||CD|=r^2$ is carried out according to the altitude theorem, since the $\triangle KCI$ is right-angled. With (*) finally follows

$$\mathcal{F}(ABC)^2=|CH|^2r^2=|CH||HB||BD||DC| \quad (**)$$

If you set $|CH|=\frac{1}{2}U=s$, then successively follows

$$|HB|=|CH|-|CB|=s-a$$

$$|DC|=|CE|=|AC|-\underbrace{|AE|}_{|HB|}=b-(s-a)=a+b-s=2s-c-s=s-c$$

$$|BD|=|BF|=|BA|-\underbrace{|AF|}_{|HB|}=c-(s-a)=a+c-s=2s-b-s=s-b$$

In total, one has obtained

$$|\mathcal{F}(ABC)|^2=s(s-a)(s-b)(s-c)$$

Outlook: Brahmagupta's formula.

The Heron formula is a special case of Brahmagupta's formula for cyclic quadrilaterals

$$\mathcal{F}=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

It should be mentioned that Brahmagupta generally uses the formula for quadrilaterals without mentioning the condition of the circumcircle. In the limit case $d\to 0$ the Heron formula is obtained. A cyclic quadrilateral with an integer area is also called a Brahmagupta quadrilateral. Such a quadrilateral has the sides $a=65$, $b=25$, $c=33$, $d=39$. The semi-circumference is $s=\frac{1}{2}(65+25+33+39)$. The area is therefore

$$\begin{aligned}\mathcal{F}&=\sqrt{(81-65)(81-25)(81-33)(81-39)}\\&=\sqrt{16\cdot 56\cdot 48\cdot 42}=\sqrt{1806336}=1344\end{aligned}$$

Even the diagonals are integers

$$e^2=\frac{(ac+bd)(ad+bc)}{ab+cd}=\frac{3120\cdot 3360}{2912}=3600\Rightarrow e=60$$

$$f^2 = \frac{(ab+cd)(ac+bd)}{ad+bc} = \frac{2912 \cdot 3120}{3360} = 2704 \Rightarrow f = 52$$

The radius of a circumcircle is not necessarily an integer

$$R = \frac{1}{4A}\sqrt{(ab+cd)(ac+bd)(ad+bc)} = \frac{1}{4 \cdot 1344}\sqrt{2912 \cdot 3120 \cdot 3360} = \frac{65}{2}$$

Another Brahmagupta quadrilateral has the sides (52, 25, 39, 60). Brahmagupta's formula is a special case of the area of general (convex) quadrilateral. If β and δ are the opposite angles of the quadrilateral, then applies the formula by C.A. Bretschneider[19]

$$\mathcal{F} = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd \cdot \left[\cos\left(\frac{\beta+\delta}{2}\right)\right]^2}$$

For opposite inner angles of cyclic quadrilaterals follows, $\beta + \delta = 180°$; thus provides $\cos\left(\frac{\beta+\delta}{2}\right) = 0$, so the term goes back into Brahmagupta's formula.

17.5 Cube Doubling by Heron

W.R. Knorr ascribes the following Neusis construction for cube doubling, which can be found in the *Metrica*, to Heron himself, as well as Eutokios and Pappos doo. However, Knorr denies Heron any originality, since he allegedly took over most of his works from Archimedes. So the construction could ultimately come from Archimedes.

The designations of Fig. 17.8 apply. We restrict ourselves here to the construction of $\sqrt[3]{2}$, so that one can set $\frac{|AB|}{|AC|} = 2$. This determines the ratio of sides of rectangle $ABCD$ and the diagonal point E. The line segment $|BF| = x = \sqrt[3]{2}$. is sought. For this purpose, a point F at the straight line AB and a point G at the line AC are moved until both $|EF| = |EG|$ is reached and at the same time vertex D lies on the line FG. So F and G lie on a circle with center E.

Proof: It is to be shown that the line $|BF|$ has the desired length. Because the triangles $\triangle BFD$ and $\triangle CDG$ are similar, it follows that $\frac{y}{2} = \frac{1}{x} \Rightarrow y = \frac{2}{x}$. Because of $|EF| = |EG|$, Pythagoras applies and it follows that

$$(x+1)^2 + \left(\frac{1}{2}\right)^2 = \left(y + \frac{1}{2}\right)^2 + 1^2 \Rightarrow x^2 + 2x = y^2 + y$$

, After simplifying and factoring out, it holds

[19] Bretschneider C.A.: *Untersuchungen der trigonometrischen Relationen des geradlinigen Vierecks,* Archiv d. Mathematik und Physik, Greifswald 1842, S. 225.

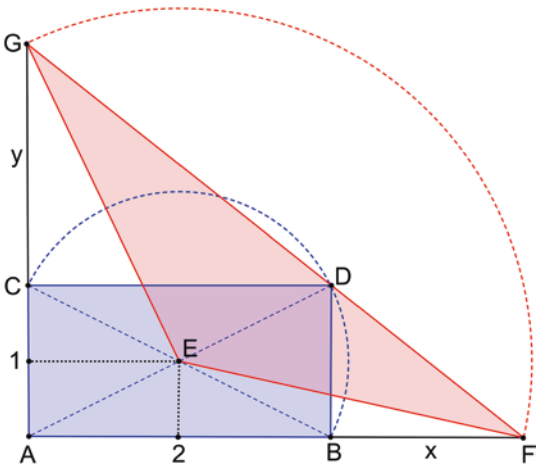

Abb. 17.8 Figure for cube doubling

$$x^4 + 2x^3 = 2x + 4 \Rightarrow x^3(x+2) = 2(x+2)$$

Since $x = -2$ is no solution, it can be simplified to $x^3 = 2 \Rightarrow x = \sqrt[3]{2}$.

17.6 Area of the Regular Pentagon

As an example of a slightly more involved calculation, the area determination of a regular pentagon from the *Geometrica* is treated here. For the construction of a partial triangle *BME* of the regular pentagon, Heron starts from the following construction (Fig. 17.9).

Given is the right-angled triangle *ABC* with a right angle at *A* and the angle $\frac{2}{5}R$ at *C*. The reflection of *C* at *A* yields the point *M*, the reflection of *B* at *A* yields the point *E*. *M* is the center of the circle with the radius $r = |BM| = |EM|$. Producing of *EB* yields the point of intersection *D* with the circle.

Proof: $\triangle CBM$ is isosceles due to the reflection at *A*. Thus, the angle $\measuredangle BMA$ is also $\frac{2}{5}R$ and $\measuredangle MBA = \frac{3}{5}R$. Since $\measuredangle DBC$ is a straight angle, it follows that $\measuredangle DBM = 2R - 2 \cdot \frac{3}{5}R = \frac{4}{5}R$. Since $\triangle DMC$ is isosceles, the angle at the $\measuredangle DMC = \frac{2}{5}R$; $\triangle DMC$ is therefore a partial triangle of the regular decagon; accordingly, $\triangle BME$ is also a partial triangle of the regular pentagon. $\triangle DMC$ is therefore also isosceles and has the property of [Euclid IV, 10]. Is a base angle twice as large as the angle at the apex *C*; the line *CD* is divided in extreme and mean ration at point *B*. According to [Euclid XIII, 1] it follows that $(|CB| + |AC|)^2 = 5|AC|^2$. This can be seen by a small side calculation. If $|CD| = |CM| = a$, then because of the just mentioned division it holds $|CB| = \frac{a}{2}\left(\sqrt{5} - 1\right)$. Thus

$$|CB| + |AC| = \frac{a}{2}\sqrt{5} \Rightarrow (|AB| + |AC|)^2 = 5\frac{a^2}{4} = 5|AC|^2$$

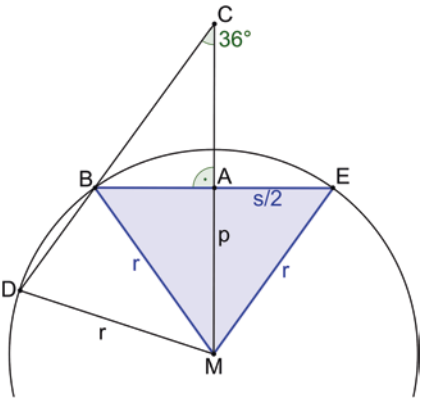

Fig. 17.9 Figure to the regular pentagon

Calculation: In the partial triangle $\triangle BME$ the following applies $p = \frac{1}{2}|CM| = |AC|$. According to Pythagoras, because of the congruence $(r+p)^2 = 5p^2$. With the Heronian approximation $\sqrt{5} = \sqrt{4+1} \approx 2 + \frac{1}{2\cdot 2} = \frac{9}{4}$ follows

$$r + p = \frac{9}{4}p \Rightarrow r = \frac{5}{4}p$$

For the side s of the pentagon the following applies: $\left(\frac{s}{2}\right)^2 = r^2 - p^2 = \frac{25}{16}p^2 - p^2 = \frac{9}{16}p^2 \Rightarrow \frac{s}{2} = \frac{3}{4}p \Rightarrow p = \frac{2}{3}s$

The area of $\triangle BME$ is $\frac{1}{2}sp = \frac{1}{3}s^2$; so the desired area of the regular pentagon is nearly $A \approx \frac{5}{3}s^2$. For a better root approximation than $\frac{9}{4}$, Heron gives the result $A \approx \frac{12}{7}s^2$.

The modern value is $A = \frac{1}{4}s^2\sqrt{25+10\sqrt{5}} \approx 1{,}720s^2$.

17.7 Root Calculation Among the Greeks

Babylonian Root Calculation

The Babylonian root calculation is based on the formula by K. Hunrath[20]

$$a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1} (b \ll a)$$

The proof of the upper limit is carried out by squaring

$$\left(a \pm \frac{b}{2a}\right)^2 = a^2 \pm b + \underbrace{\left(\frac{b}{2a}\right)^2}_{\approx 0}$$

[20] Hunrath K.: Über das Ausziehen der Quadratwurzel bei den Griechen und Indern, Schütze & Festersen, 1883.

Heron was familiar with the left side of the above root approximation, as can be seen from his work *Metrica* (I, 8); the right side can be found in al-Kharki in the eleventh century. Examples from Heron's calculations are

$$\sqrt{10} = \sqrt{3^2 + 1} \approx 3 + \frac{1}{6} = \frac{19}{6}$$

$$\sqrt{4500} = \sqrt{67^2 + 11} \approx 67 + \frac{11}{134} = \frac{8989}{134}$$

$$\sqrt{2} = \sqrt{\left(\frac{3}{2}\right)^2 - \frac{1}{4}} \approx 1{,}5 - \frac{\frac{1}{4}}{2 \cdot 1{,}5} = 1{,}5 - \frac{1}{12} = \frac{17}{12}$$

The value of $\sqrt{2}$ passed down by the Indians can also be explained by this

$$\sqrt{288} = 12\sqrt{2} = \sqrt{17^2 - 1} \approx 17 - \frac{1}{2 \cdot 17}$$
$$\Rightarrow \sqrt{2} \approx \frac{17}{12} - \frac{1}{12 \cdot 34} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}$$

If an approximation is not precise enough, the process can be iterated

$$\sqrt{3} = \sqrt{\left(\frac{9}{5}\right)^2 - \frac{6}{25}} \approx \frac{9}{5} - \frac{\frac{6}{25}}{2 \cdot \frac{9}{5}} = \frac{26}{15}$$

In the second step, this approximation value is followed by

$$\sqrt{3} = \sqrt{\left(\frac{26}{15}\right)^2 - \frac{1}{225}} \approx \frac{26}{15} - \frac{\frac{1}{225}}{2 \cdot \frac{26}{15}} = \frac{1351}{780}$$

The iteration here provides the upper bound of the Archimedes approximation for $\sqrt{3}$. T. Heath (II, p. 52) provides another proposal for the Archimedes bounds. He uses the following transformations

$$\sqrt{675} = 15\sqrt{3} = \sqrt{26^2 - 1}$$
$$\Rightarrow 26 - \frac{1}{51} < 15\sqrt{3} < 26 - \frac{1}{52}$$
$$\Rightarrow \frac{1351}{780} < \sqrt{3} < \frac{265}{153}$$

H. W. Turnbull[21] recognizes the Archimedes bounds in the chain fractions and believes that the corresponding knowledge can be assumed

[21] Turnbull H.W., The Great Mathematicians, New York University Press 1929, p. 41.

$$\frac{1}{3}\left(5+\frac{1}{5+\frac{1}{10+\frac{1}{5}}}\right)<\sqrt{3}<\frac{1}{3}\left(5+\frac{1}{5+\frac{1}{10}}\right)$$

J. E. Hofmann[22] suggests the following root approximation

$$\sqrt{a^2-1}\approx a-\frac{1}{2a-1}+\frac{1}{(2a-1)(2a+1)}$$

An example is

$$\sqrt{3}=\sqrt{2^2-1}\approx 2-\frac{1}{4-1}+\frac{1}{(4-1)(4+1)}=2-\frac{1}{3}+\frac{1}{15}=\frac{26}{15}$$

Heron gives his fractions in the hexagesimal system. An example is $\sqrt{2}=\underbrace{1+\frac{24}{60}}_{1,4}+\frac{51}{60^2}$. In decimal, this is $\sqrt{2}=\frac{1697}{1200}\approx 1{,}41416$. If you set (as above) the difference $\sqrt{2}-1{,}4=\frac{1}{70}$ equal to $\frac{x}{3600}$, then $x=51$. The specified approximation is therefore correct to $\frac{1}{3600}$.

But Heron also provides square roots with the accuracy $\frac{1}{60^3}$„ for example

$$\sqrt{27}=5+\frac{11}{60}+\frac{46}{60^2}+\frac{50}{60^3}=5\frac{4241}{21600}$$

Because of $\left(5\frac{4241}{21600}\right)^2=27{,}001976$, the deviation is greater than expected with the accuracy $\frac{1}{60^3}$. In fact, it must be said

$$\sqrt{27}=511'46''10'''$$

Heron calculates the cube root $\sqrt[3]{100}$ and seems to use the following approximation formula. If N is the radicand of the cube root with the bounds $a^3<N<b^3$, then

$$\sqrt[3]{N}\approx a+\frac{bc}{bc+ad}(b-a)\ mit\ c=N-a^3, d=b^3-N.$$

For the example, this results in

$$4^3<100<5^3\Rightarrow\sqrt[3]{100}=4+\frac{5\cdot 36\cdot(5-4)}{5\cdot 36+4\cdot 25}=4\frac{9}{14}$$

The mathematical background of this formula is discussed in T. Heath [II, p. 341].

Heron's Method

The algorithm of Heron uses for $\sqrt{a}$ at an approximation value $x>0$ as a starting value the formula

[22] Hofmann J.E.: Quadratwurzel bei Archimedes und Heros, in the collected volume Becker (1985), p. 112.

$$y = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

This approach can be derived from the Babylonian method. If x is the approximation value of the root $\sqrt{a}$, then with $b = a - x^2$

$$\sqrt{a} = \sqrt{x^2 + b} \approx x + \frac{b}{2x} = \frac{2x^2 + a - x^2}{2x} = \frac{x^2 + a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

In American literature, the two methods are sometimes not distinguished here. For $\sqrt{2}$, with the starting value $x = \frac{7}{5}$, which Plato already knew

$$\sqrt{2} \approx \frac{1}{2}\left(\frac{7}{5} + \frac{2}{\frac{7}{5}}\right) = \frac{99}{70}$$

If the achieved accuracy is not sufficient, the method can be iterated by using the previously obtained approximation as the new starting value

$$\sqrt{2} \approx \frac{1}{2}\left(\frac{99}{70} + \frac{2}{\frac{99}{70}}\right) = \frac{19601}{13860} = 1{,}41421356$$

This already gives 9 valid digits!
The sequence resulting from iteration converges (in second order) to $\sqrt{a}$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

This shows the fixed point equation

$$x = \frac{1}{2}\left(x + \frac{a}{x}\right) \Rightarrow x = \frac{a}{x} \Rightarrow x^2 = a$$

The boundedness of the monotonically decreasing sequence is seen using the inequality of the arithmetic and geometric mean

$$\frac{1}{2}\left(x + \frac{a}{x}\right) \geq \sqrt{x \cdot \frac{a}{x}} = \sqrt{a}$$

This ensures the convergence (with a suitable starting value).

17.8 Other Works by Heron

In addition to his work in mathematics, Heron was particularly an engineer and machine builder. Figure 17.10 shows Heron demonstrating his steam boiler in front of a group of engineers. In addition to the works afore mentioned , the following are recognized as pseudo-heronian:

- Geometrica
- Stereometrica

Fig. 17.10 Heron demonstrates his steam boiler in Alexandria, AKG3119985 Copyright akg/ Science Photo Library

- Mensurae
- Metrica,
- Fire weapons *(Cheiroballistra)*
- Definitiones

For a discussion of the assignment of the pseudo-Heron works, the framework is exceeded here, a detailed discussion can be found with H. Schellenberg (see footnote). Pappus calls him in his book VIII a *mechanic*. He writes

> The mechanics of Heron's school say that mechanics can be divided into a theoretical and a manual part. The theoretical part consists of geometry, arithmetic, astronomy and physics, metals, architecture; the other with carpentry and painting as well as everything that has to do with craftsmanship.
>
> The ancients also call the miracle workers mechanics, some of whom work with pneumatics, *as* Heron in his *Pneumatica*, some using threads and ropes to simulate the movements of living beings, *as* Heron in his automata and balances, […] where he uses water to determine the time, *as* Heron in his *Hydria*, which seems to work similarly to the sundial.

A) Example of Catoptrica

In the *Dioptra* (optics) he describes devices for surveying; the dioptra itself is an instrument that fulfilled the function of today's theodolites.

In the *Catoptrica*, Heron sets forth the optical principle: The path of light between two points always proceeds such that the path of light is a minimum. Although this belongs to physics, it is to be presented here because its justification proceeds geometrically.

Given are two points P, Q on the same side of a line AB, but not on the line itself. Sought is the point R of the line such that the path from P over R to Q is the shortest: $|PR| + |RQ| \rightarrow$ *Minimum* (Fig. 17.11).

First, point P is reflected at line AB; reflection point is P'. The point of intersection of line AB with the ray $P'Q$ is R. To be shown is that R is the sought point.

Contradiction Proof

We assume that the point R' is the sought point of the shortest path. Because of the law of reflection, it follows: $|PR| = |P'R|$ and $|PR'| = |P'R'|$. For the light path over R it follows $|PR| + |RQ| = |P'R| + |RQ| = |P'Q|$. For the light path over R' :q

$$|PR'| + |R'Q| = |P'R'| + |R'Q|$$

In the triangle $\triangle P'R'Q$ applies: The sum of the sides $|P'R'| + |R'Q|$ is greater than the third side $|P'Q|$; thus, $|PR'| + |R'Q| > |P'Q|$. From this it follows $|PR'| + |R'Q| > |PR| + |RQ|$; this is a contradiction to the assumption. Thus, the claim is true: $|PR| + |RQ| \rightarrow$ *Minimum*.

$\measuredangle QRB$ is congruent to $\measuredangle PRA$ as the opposite angle, because of the reflection $\measuredangle PRA$ is congruent to $\measuredangle P'RA$. Thus, the angles are also congruent $\measuredangle PRA = \measuredangle QRB$. R is therefore the point at which the angle of incidence and reflection are congruent.

B) To Pneumatics

In the preface of *Pneumatics* it is stated that Heron:

> By the combination of air, earth, fire and water and the interaction of three or four basic principles, various effects are achieved, some of which serve to fulfill the most urgent desires of daily life, while others cause amazement and consternation.

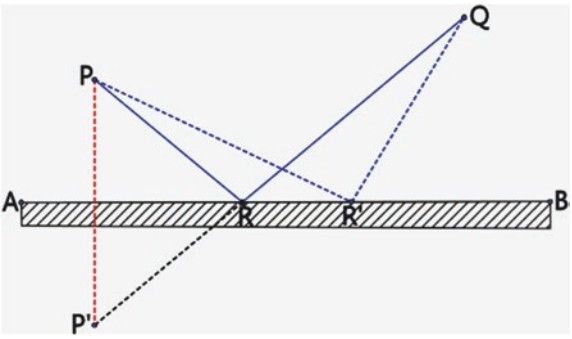

Fig. 17.11 Figure for the minimal path of light

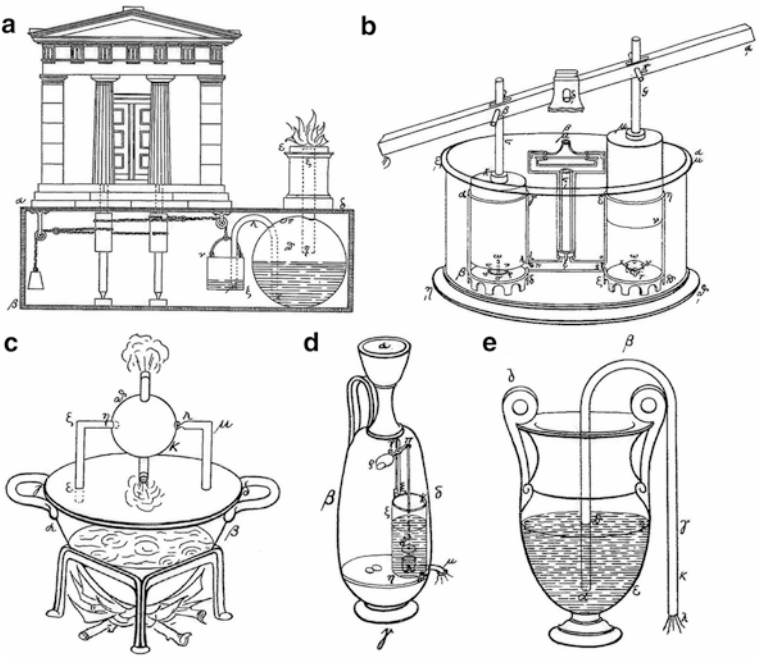

Fig. 17.12 Machines and automats Herons

A variety of inventions are described in this work, e.g. the construction of a holy water dispenser (d). After inserting a coin, a certain amount of holy water was released. With the Heron ball (c), a fire-operated, water vapor-filled, rotatable hollow ball with tangential steam outlet, Heron provides the first known and documented heat engine in history. Important inventions were the fire pump (b) and the suction pump (e). All machines and devices of Heron can be found in the anthology Schmidt[23].

C) To the Work Automata

In his work *automata* (automaton construction) there is an apparatus as Automaton No. 73, which causes the temple doors to open automatically after a fire is lit on the temple forecourt (a). In addition to the mentioned machines, Heron also developed music automatons; for the believers in the temples the already mentioned holy water automaton. For the Alexandrians, the experience of these performances must have had a sensational effect at that time.

All mentioned machines are shown in Fig. 17.12. Ktesibios of Alexandria was the predecessor in automaton construction, who built a water clock (compare

[23] Heronis Alexandrini Opera Quae supersunt omnia, Schmidt. W.(Hrsg.):Vol. I, Teubner 1976.

Fig. 16.9) and a water organ, and his presumed pupil, *Philon* of Byzantium, who worked in Alexandria and Rhodes. The latter is the author of the idea of a self-filling oil lamp, he also knew the principle of the siphon.

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Claudius Ptolemy

18

I know well that I am mortal, a creature of one day,
but when my mind contemplates the stars,
the circling course of their intertwined paths,
then my feet no longer touch the earth,
at Zeus' side ambrosia, the food of the gods, refreshes me.

(Claudius Ptolemy, Greek Anthology IX, 577)

Claudius Ptolemy (Fig. 18.1) probably came from Ptolemais Hermii (Egypt), as reported by a Byzantine source. The Suda Lexicon calls him Κλαύδιος Πτολεμαῖος. He worked as an astronomer and geographer in Alexandria. Since *Claudius* is a Latin name, it can be assumed that he or a male ancestor became a Roman citizen, possibly after being freed. He was probably born around 100 AD and died around 170 AD; at any rate, his life span extended into the relatively peaceful reigns of the emperors Trajan (98–117 AD), Hadrian (117–138 AD), Antoninus Pius (138–161 AD) and Marcus Aurelius (161–180 AD). It is possible that Theon of Smyrna was his teacher, since he quotes him several times. Even an epigram by him has been handed down (see above).

The only certain points of reference for his lifetime arise from the observations recorded in his astronomical main work: The earliest refers to a lunar eclipse from April 125 AD, the latest to a Mercury elongation from February 141 AD. According to a report by Olympiodoros, he moved his observatory to Kanopus (15 km east) in the last 20 years and later erected a stele there, in which the improved parameters of the planetary orbits (compared to the Almagest) were chiseled in (Fig. 18.2); the mentioned Kanopus inscription can be dated to 147/148 AD.

His most important work is the 8-book work *Almagest* (see references: Manitius (ed.)), which was probably completed after 141 BC. It had decisive importance in astronomy until the publication of Copernicus' *De Revolutionibus Orbium Coelestium,* which placed the sun in the center of the planetary system. The old geo-centric world view valid until then is inextricably linked with the name *Ptolemaic.*

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_18

Fig. 18.1 Claudius Ptolemy (Wikimedia Commons)

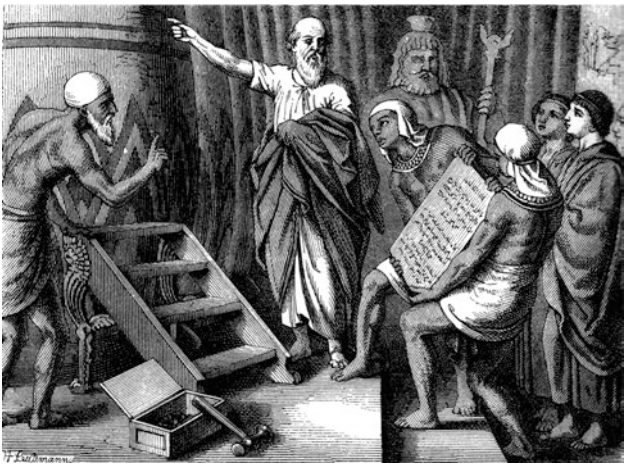

Fig. 18.2 Ptolemy erects a plaque with astronomical data in Kanopus

Some authors deny him any originality, because he took over very many star positions from Hipparchus. As early as 1819, the French astronomer J. Delambre had recognized that Ptolemy had falsified some solar values. The American R. Newton[1]

[1] Robert R. Newton, *The Crime of Claudius Ptolemy,* Baltimore 1977.

accused him of complete fraud in 1977; he had not measured a single astronomical value independently. Also B.L. van der Waerden[2] writes in 1988 that

> Ptolemy had systematically and deliberately falsified observations to bring them into line with his theory, as Newton and Delambre had shown.

G.J. Toomer writes:

> As a didactic work, the Almagest is a masterpiece of clarity and methodology, superior to any ancient textbook and with very few peers in all times. But it is more than that. Far from being merely a systematization of early Greek astronomy, as it is sometimes described, it is in many respects an original work.

A precise and objective analysis that includes a complete error analysis can be found in G. Grasshoff; he notes:

> It must be assumed that a substantial part of the Ptolemaic star catalogue is based on observations by Hipparchus, which Hipparchus already used for the second part of his commentary on Aratus. […] The processing of Hipparchus' observational values should no longer be considered from the perspective of plagiarism. Ptolemy, whose intention was to create a comprehensive theory of celestial bodies, had no access to the methods of modern data analysis. […] For methodological reasons, Ptolemy was forced to choose the value from a series of observations that was, in his opinion, the most reliable value. Ptolemy had to regard such values as "observed" that could be confirmed by theoretical considerations.

The Almagest consists of 8 books. Book I contains the mathematical foundations, books II–VII the listing of the known places of the *Oikumene.* Book VIII deals with astronomical questions, hints for the creation of maps and the table of the longest day (at the summer solstice) depending on the latitude. A detailed treatment of Ptolemy's astronomical writings exceeds the scope of this book. A very good description of the astronomical work and its long reception history is given by G. Grasshoff[3]; there one finds 50 maps of the most important constellations (p. 219–269) and the catalogue of 1028 star coordinates (p. 275–315), each with comparison to the modern values.

18.1 Trigonometry in the Almagest

In the context of this book, we are mainly concerned with his mathematical work Ματηεματικε Σψνταξις.. Since his *calculations of chords inscribed in a circle* (now part of Book I) were significant for astronomical purposes, the work was translated into Arabic in 827 and revised at the court of al-Ma'mun, among others

[2] B. L. van der Waerden, *Die Astronomie der Griechen,* Wissenschaftliche Buchgesellschaft 1988, S. 253.

[3] Grasshoff G.: The History of Ptolemy's Star Catalogue, Springer 1990.

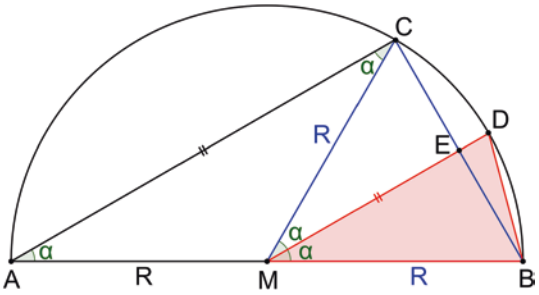

Fig. 18.3 Conversion of cord to sine of an angle

by ibn Thābit and al-Ḥağğağ. There it was first called Μεγαλε Σψνταξις *(Great Collection)* and was finally called *Tabrir al mageste*, where the word *magele* was transformed into *mageste* by intensification. The *Almagest* was the determining trigonometry book until modern times. The first chord table in Latin *De triangulis omnimodis* was written by Regiomontanus in 1464; it was not printed until 1533 posthumously.Since, for a given circle radius, a chord length can be converted to the sine of the corresponding center angle, a cord table represents the beginning of *trigonometry*. As with the star coordinates of Almagest, Ptolemy borrows heavily from his predecessor Hipparchus (around 140 BC).

Conversion of Cords to Sine Values

Since Ptolemy calculates fractions in the Babylonian sexagesimal system, he sets the unit (i.e., the circle radius R) to 60 units. For the angles he adopts the 360° -scale. The relationship is (Fig. 18.3):

$$\sin\alpha = \frac{|BE|}{|MB|} = \frac{2|BE|}{2|MB|} = \frac{|BC|}{|AB|} = \frac{\mathrm{crd}(2\alpha)}{2R} \Rightarrow \mathrm{crd}(2\alpha) = 2R\sin\alpha$$

crd is here the abbreviation for *chord* (Eng. *cord*).

Special Angle Values

The starting point of his tendon calculation were the angles that are determined by known three-, five- and ten-sided figures. The isosceles or right-angled isosceles triangle provides the values:

$$\mathrm{crd}(60°) = R = 60$$

$$\mathrm{crd}(90°) = \sqrt{2}R = 60\sqrt{2} = 84 + \frac{51}{60} + \frac{10}{60^2}$$

$$\mathrm{crd}(120°) = \sqrt{3}R = 103 + \frac{55}{60} + \frac{23}{60^2}$$

The penultimate result shows that Ptolemy used the following approximation: $\sqrt{2} = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$. According to Euclid XIII, 10, the side of the regular decagon:

$$s_{10} = |ZD| = |ZE| - |DE| = |BE| - |DE| = \sqrt{R^2 + \left(\frac{R}{2}\right)^2} - \frac{R}{2} = \frac{R}{2}\left(\sqrt{5} - 1\right)$$

This distance is the tendon to the 36° -angle, thus follows

$$\text{crd}(36^\circ) = \frac{R}{2}\left(\sqrt{5} - 1\right) = 30\left(\sqrt{5} - 1\right) = 37 + \frac{4}{60} + \frac{55}{60^2}$$

For the side of the regular pentagon follows analogously:

$$s_5 = |BZ| = \sqrt{|ZD|^2 + |BD|^2} = \sqrt{(s_{10})^2 + R^2} = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}$$

This is the chord to the 72° -angle, thus

$$\text{crd}(72^\circ) = 30\sqrt{10 - 2\sqrt{5}} = 70 + \frac{32}{60} + \frac{3}{60^2}$$

Angle Halving

As can be seen from Fig. 18.3, for the tendon of the supplement angle $(180^\circ - \alpha)$ according to Pythagoras

$$\text{crd}(180^\circ - 2\alpha) = |AC| = \sqrt{4R^2 - |BC|^2} = \sqrt{4R^2 - [\text{crd}(2\alpha)]^2}$$

Ptolemy gives the following example for this

$$\text{crd}(144^\circ) = \text{crd}(180^\circ - 36^\circ) = \sqrt{4R^2 - [\text{crd}(36)]^2} = \sqrt{14.400 - \left(37 + \frac{4}{60} + \frac{55}{60^2}\right)^2}$$

$$\Rightarrow \text{crd}(144^\circ) = \sqrt{13.024 + \frac{55}{60} + \frac{45}{60^2}} = 114 + \frac{7}{60} + \frac{37}{60^2}$$

The chord to theangle at the center 2α is $|BC|$, to the angle $\alpha\,|BD|$. Thus, it follows that

$$[\text{crd}(\alpha)]^2 = |BD|^2 = |BE|^2 + |ED|^2 = \frac{1}{4}|BC|^2 + |ED|^2 = \frac{1}{4}[\text{crd}(2\alpha)]^2 + |ED|^2$$

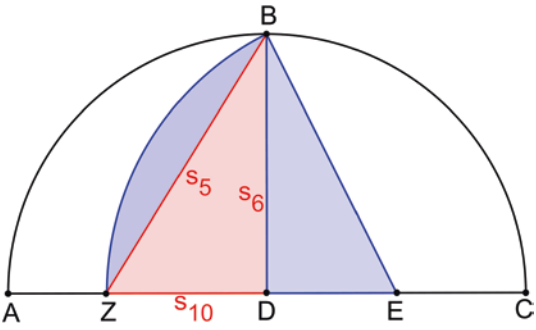

Fig. 18.4 Figure to the side of the decagon

In this case,

$$|ED| = |MD| - |ME| = R - \frac{1}{2}|AC| = R - \frac{1}{2}\sqrt{4R^2 - [\text{crd}(2\alpha)]^2}$$

With these formulas, Ptolemy succeeds in the chord calculation with continued angle halving $60° \to 30° \to 15° \to 7{,}5° \to 3{,}75°$ or $72° \to 36°$.

The Formula for 1°

By means of the chord formula for angular differences, he can calculate the chords from $6° = 36° - 30°$ by halving to 1,5° and 0,75°. He receives

$$\text{crd}(1{,}5°) = 1 + \frac{34}{60} + \frac{15}{60^2} \therefore \text{crd}(0{,}75°) = \frac{47}{60} + \frac{15}{60^2}$$

For interpolation of the intermediate values, he uses the monotonicity law already discussed

$$\beta > \alpha \Rightarrow \frac{\text{crd}(\beta)}{\text{crd}(\alpha)} < \frac{\beta}{\alpha}$$

Insertion of the value pair $\alpha = 1, \beta = \frac{3}{2}$ yields

$$\text{crd}(1{,}5°) < \frac{3}{2}\text{crd}(1°) \Rightarrow \text{crd}(1°) > \frac{2}{3}\text{crd}(1{,}5°) = 1 + \frac{2}{60} + \frac{50}{60^2}$$

Insertion of the value pair $\alpha = \frac{3}{4}, \beta = 1$ results in

$$\text{crd}(1°) < \frac{4}{3}\text{crd}(0{,}75°) \Rightarrow \text{crd}(1°) < 1 + \frac{3}{60}$$

The chord crd(1°) is thus sufficiently well determined. This is shown in the calculation of the number of circles π:

$$\pi = \frac{U}{2R} = \frac{360}{120}\text{crd}(1°) = 3\left(1 + \frac{2}{60} + \frac{50}{60^2}\right) = 3 + \frac{8}{60} + \frac{30}{60^2} \approx 3{,}1417$$

This value, as Ptolemy notes in Book (VI, 7), lies in the interval given by Archimedes

$$3\frac{10}{71} < \frac{377}{120} < 3\frac{1}{7}$$

At the same time, he mentions that this approximation is quite close to the mean of $3\frac{10}{71}$ and $3\frac{1}{7}$.

For smaller angles, linear interpolation can be used, for example

$$\text{crd}(0{,}5°) = \frac{1}{2}\text{crd}(1°) = \frac{31}{60} + \frac{25}{60^2}$$

For the first 10 values of the tendon table, the corresponding sine values are also given; the complete table can be found in Book (I, 11) of the Almagest.

α	crd(α)	Decimal	$2R \sin \frac{\alpha}{2}$
0,5°	$\frac{31}{60} + \frac{25}{60^2}$	0,5236	0,5236
1°	$1 + \frac{2}{60} + \frac{50}{60^2}$	1,0472	1,0472
1,5°	$1 + \frac{34}{60} + \frac{15}{60^2}$	1,5708	1,5708
2°	$2 + \frac{5}{60} + \frac{40}{60^2}$	2,0944	2,0943
2,5°	$2 + \frac{37}{60} + \frac{4}{60^2}$	2,6178	2,6178
3°	$3 + \frac{39}{60} + \frac{52}{60^2}$	3,1411	3,1412
3,5°	$3 + \frac{39}{60} + \frac{52}{60^2}$	3,6644	3,6646
4°	$4 + \frac{11}{60} + \frac{16}{60^2}$	4,1878	4,1879
4,5°	$4 + \frac{42}{60} + \frac{40}{60^2}$	4,7111	4,7112
5°	$5 + \frac{14}{60} + \frac{4}{60^2}$	5,2344	5,2343

According to a reconstruction by G.J. Toomer, the table of chords of Hipparchus had a larger step width 7,5°, but in return a higher accuracy of $\frac{1}{60^3}$. An application of the chordal calculation is given by Ptolemy in the Geography (Fig. 18.5). He determined the latitude of Rhodes to 36° and calculated the radius of the corresponding parallel of the latitude as crd(108°).

Without giving the exact calculation, he finds the value

$$crd(108°) = \frac{93}{115} \approx 0{,}80870\ (R_{\varphi})$$

Actually, one would expect the denominator 120 here. Toomer explains this with the fact that Ptolemy here uses a different table of chords, probably that of

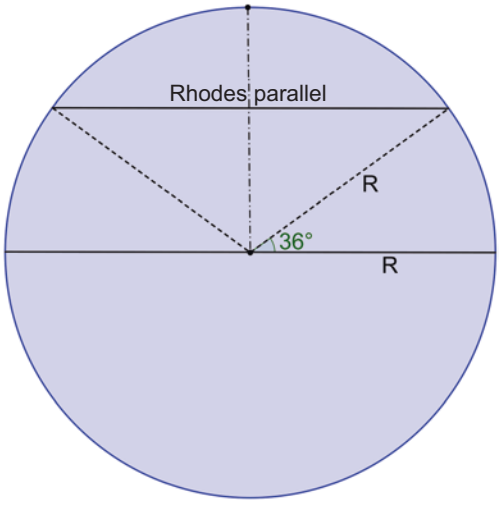

Fig. 18.5 Figure for calculating the parallel of the latitude

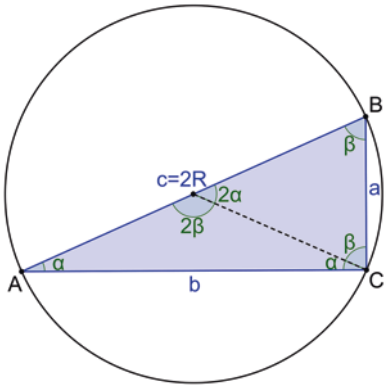

Fig. 18.6 Figure for the right-angled triangle

Hipparchus. The precise value can be calculated using the radius r of the parallel circle

$$\frac{r}{R} = \cos 36° \Rightarrow \operatorname{crd}(108°) = \cos 36° = \frac{1}{4}\left(1+\sqrt{5}\right) \approx 0{,}80902\,(R_{\varphi})$$

It is curious that up until the seventeenth century, the theorem of Menelaus was attributed to Ptolemy because in Book (I, 13) of the Almagest he provides a proof without mentioning Menelaus. Only when Father M. Mersenne (1588–1648) discovered a manuscript of Menelaus the attribution could to be corrected.

Calculation of Right-Angled Triangles

a) Given is the side ratio $\frac{a}{c} = \frac{7}{9}$. According to Fig. 18.6 follows

$$\frac{a}{c} = \frac{\operatorname{crd}(2\alpha)}{2R} \Rightarrow \operatorname{crd}(2\alpha) = 2R\frac{a}{c} = 120 \cdot \frac{7}{9} = 93{,}33$$

By interpolation, read from the chord table: $2\alpha = 102{,}12° \Rightarrow \alpha = 51{,}06°$. The second angle is then $\beta = 38{,}94°$.

b) Given angle $\alpha = 28{,}07°$ follows by interpolation $\operatorname{crd}(2\alpha) = 56{,}466$. With the above formula the corresponding side is

$$a = c\frac{\operatorname{crd}(2\alpha)}{2R}$$

Given hypotenuse $c = 17$ follows $a = 17 \cdot \frac{56{,}466}{120} = 8$. The missing side is then $b = \sqrt{17^2 - 8^2} = 15$.

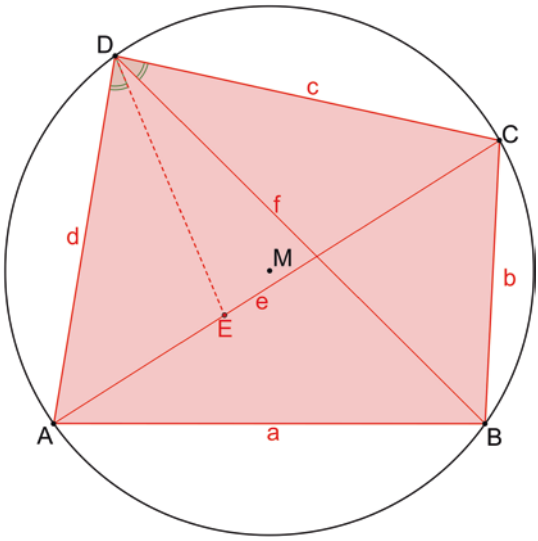

Fig. 18.7 Proof figure for Ptolemy's theorem

18.2 Ptolemy's Theorem

Ptolemy's theorem is stated in book (I, 10); it states that in the cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides. Here thus

$$ef = ac + bd$$

Ptolemy's Proof
As an auxiliary line he uses the line DE, which results applying the angle $\measuredangle BDC$ to side AD. Thus $\measuredangle ADE$ and $\measuredangle BDC$ are congruent (Fig. 18.7).

1. Claim: $\triangle ABD$ similar to $\triangle ECD$. The angles $\measuredangle ACD$ and $\measuredangle ABD$ are congruent as angles at the common segment AD. The angles $\measuredangle ADB$ and $\measuredangle EDC$ are congruent, since they are composed of the congruent angles $\measuredangle ADE$ and $\measuredangle BDC$ and the common angle EDB. Thus, the claim (1) applies.
2. Claim: $\triangle$ ADE similar to $\triangle$ DBC. The angles $\measuredangle CAD$ and $\measuredangle CBD$ are congruent as angles at the common segment CD. The angles $\measuredangle ADE$ and $\measuredangle BDC$ are congruent according to assumption. Thus, the claim (2) applies.

From claim (1) it follows:

$$\frac{|BD|}{|AB|} = \frac{|CD|}{|EC|} \Rightarrow |BD| \cdot |EC| = |AB| \cdot |CD| \quad (3)$$

From claim (2) it follows:

$$\frac{|AD|}{|AE|} = \frac{|BD|}{|CB|} \Rightarrow |AD| \cdot |CB| = |AE| \cdot |BD| \quad (4)$$

For the product of the diagonals, (3) and (4) apply

$$ef = |AC| \cdot |BD| = (|AE| + |EC|) \cdot |BD| = |AD| \cdot |CB| + |AB| \cdot |CD| = bd + ac$$

Thus, the theorem is proven.

Special Quadrilaterals

a) If the cyclic quadrilateral is a rectangle with sides a, b and the diagonal e, then Ptolemy's theorem passes into Pythagorean : $a^2 + b^2 = e^2$.
b) If the cyclic quadrilateral is a trapezoid with parallel sides a, c and the leg b, the diagonal $e = \sqrt{b^2 + ac}$ results.
c) If you put two right-angled triangles with the same hypotenuse together, you get a quadrilateral in which the common hypotenuse is also the diagonal and diameter of the circle. For the two Pythagorean triples $(15, 20, 25)$ or $(7, 24, 25)$, for example, you get the cyclic quadrilateral with $a = 15, b = 20, c = 7, d = 24$ and the circumference diameter $e = 25$. The missing diagonal f results in:

$$ef = ac + bd \Rightarrow f = \frac{ac + bd}{e} = \frac{15 \cdot 7 + 20 \cdot 24}{25} = 23{,}4$$

Reversal of the Ptolemy Theorem
The following Ptolemy Theorem can be reversed. If the equation applies to four coplanar points A, B, C and D, then the 4 points (in this order) lie on a straight line or on a circle

$$|AD| \cdot |CB| + |AB| \cdot |CD| = |AC| \cdot |BD|$$

In general, the corresponding inequality applies

$$|AD| \cdot |CB| + |AB| \cdot |CD| > |AC| \cdot |BD|$$

This theorem plays a similar role for four points as the triangle inequality for three points.

18.3 The Addition Theorem

Theorem 3 from Book I, 10 provides the addition theorem for two angles: If two arcs and the spanning chords are given, then the chord will also be given that encompasses the sum of the two arcs (Fig. 18.8).
Starting from the cyclic quadrilateral $ABCD$, which is inscribed in the semicircle over AD, Ptolemy constructs an auxiliary point A_1 so that the chord BA_1 is a congruent diameter $|AD| = |BA_1|$. The angles $\measuredangle CAD = \measuredangle CA_1D = \alpha$ or $\measuredangle BAC =$

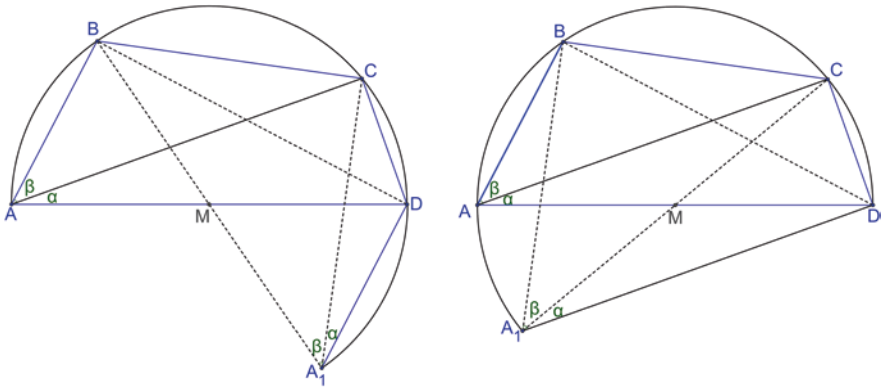

Fig. 18.8 Proof figure for the addition theorem

$\measuredangle BA_1C = \beta$ are congruent in pairs, since both are angles at the segment CD or BC. According to his theorem, in the cyclic quadrilateral BA_1DC

$$|BA_1| \cdot |CD| + |BC| \cdot |DA_1| = |BD| \cdot |CA_1|$$

A problem occurs with the string DA_1; Ptolemy has overlooked that this chord is a function of the angle $[90° - (\alpha + \beta)]$, since $\triangle BA_1C$ is right-angled according to Thales. In modern terms,

$$|DA_1| = \sin\left[90° - (\alpha + \beta)\right] = \cos(\alpha + \beta)$$

O. Neugebauer[4] solved the problem by reflecting the auxiliary point A_1 at the perpendicular bisector of AD. In the cyclic quadrilateral $BCDA$ it follows that

$$|A_1B| \cdot |CD| + |BC| \cdot |A_1D| = |BD| \cdot |A_1D|$$

If you set the radius $R = |AM| = \frac{1}{2}$, then $|AD| = 1$ and $\mathrm{crd}(2\alpha) = \sin\alpha$. As with Neugebauer, the following calculation is represented here by means of a sine function, so that the explanation is easier to read; this is *ahistorical,* since Ptolemy did not know the sine function.

In the right-angle triangle ABD you get $|BD| = \sin(\alpha + \beta)$. In the right-angled triangle A_1BC you get $|BC| = \sin\beta$ and $|A_1B| = \cos\beta$. Finally, in the right-angled triangle A_1CD $|CD| = \sin\alpha$ and $|A_1D| = \cos\alpha$. In total, the well-known Additionstheorem

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

In **Theorem 1 of Book I, 10** Ptolemy derives the corresponding theorem for the difference of angles or arcs (in modern form)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

[4] Neugebauer O.E.: History of Ancient Mathematical Astronomy, Berlin 1975, S. 23.

18.4 Construction of the Pentagon According to Ptolemy

Construction description (Fig. 18.9):

1. Draw a diameter $|FG| = 2r$
2. In the center Z construct the perpendicular diameter $|HD|$
3. Construct the center M of ZG
4. Draw a circle with center M and radius $|MD| = \frac{r}{2}$; the intersection with the line FG is I.
5. The line $|DI|$ is the side length s_5 of the regular pentagon $ABCDE$. If this line is carried out five times on the circumference (from the point D), all the corner points of the pentagon are obtained.

The length $|DZ|$ is here the radius and thus also the side of the regular hexagon s_6. The length $|IZ|$ is the side of the regular decagon s_{10}. Since the triangle $\triangle IZD$ is right-angled, it applies

$$s_{10}^2 + s_6^2 = s_5^2$$

This is the theorem [Euclid XIII, 10]. With it, the side of the regular pentagon or decagon can be easily determined.

In the unit circle, $|ZM| = \frac{1}{2}$ and $|ZD| = 1 = s_6$. In the right-angled $\triangle ZMD$ it applies

$$|DM| = \sqrt{|ZM|^2 + |ZD|^2} = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$$

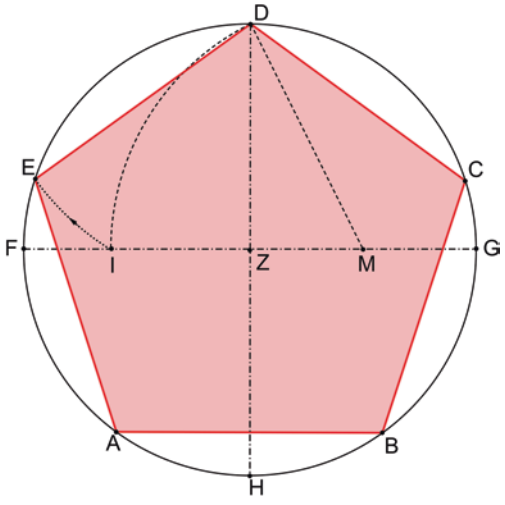

Fig. 18.9 Neusis construction of the pentagon

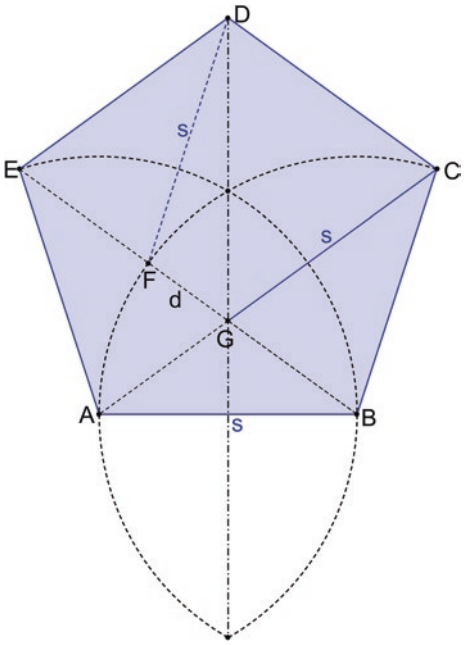

Fig. 18.10 Exact construction of the pentagon according to Ptolemy

Because of $|IZ| = \underbrace{|IM|}_{|DM|} - |ZM|$ follows $s_{10} = |IZ| = \frac{1}{2}\sqrt{5} - \frac{1}{2} = \frac{1}{2}\left(\sqrt{5} - 1\right)$. In the right-angled $\triangle IZD$ finally results in the pentagon side (in the unit circle) to

$$s_5 = \sqrt{s_{10}^2 + s_6^2} = \sqrt{\frac{1}{4}\left(\sqrt{5} - 1\right)^2 + 1} = \frac{1}{2}\sqrt{5 - 2\sqrt{5} + 1 + 4} = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}$$

Neusis Construction for Pentagon

Since the early Pythagoreans did not know the continuous division, it is to be explained how the regular pentagon was constructed. One possible solution provides here a Neusis construction, which is also later found in *A*. Dürer.

Construction description (Fig. 18.10):

Given the side length $s = |AB|$, the circles with center A resp. B are drawn through the other point; both circles intersect on the symmetry axis to AB. With the help of a ruler, on which the distance s is marked, the distance is fitted so that it fits exactly between a point (C) of the circle around B and a point (G) of the median. As a guide, the line CG can be produced to vertex A. Vertex E is the symmetrical point of C with respect to the axis of symmetry, corner point D results from the intersection of the axis and the circle with Center C (radius *s)*. The diagonal $d = |BE| = |AC|$ fulfills the criterion for the division in extreme and mean ratio (Euclid II, 11).

$$d(d - s) = s^2$$

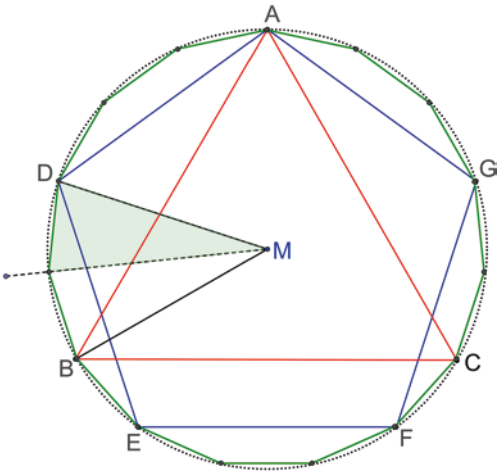

Fig. 18.11 Construction of the 15-gon

18.5 Construction of the 15-gon

The fact that the regular pentagon and triangle can be constructed also implies that the 15-gon can be constructed (Fig. 18.11). The angle $\measuredangle AMB$ at the center of the equilateral triangle is 120°; the corresponding angle of the pentagon $\measuredangle AMD$ is 72°. The angle difference $\measuredangle BMD$ is therefore 48°, halving it results in a 24° angle. This is exactly the angle at the center of the regular 15-gon.

The combination of the regular triangle and pentagon to form the regular 15-gon can be calculated algebraically (as a diophantine equation)

$$3x + 5y = 1$$

One possible solution is $x = 2$; $y = -1$. Therefore, after division by 15

$$3(2) + 5(-1) = 1 \Rightarrow \frac{2}{5} - \frac{1}{3} = \frac{1}{15}.$$

This means that the arc upon one side off the regular 15-gon results from the double arc of the regular pentagon reduced by the arc of the regular triangle. This sentence can be found with Euclid (IV, 16).

18.6 The Geographical Work

Similarly important for the history of the natural sciences—as the Almagest for astronomy—is the book *Introduction to Geography* (Γεωγραφικὴ Ὑφήγησις) by Ptolemy. There are numerous Arabic copies of this writing. In his work, Ptolemy tried to bring the coordinates of about 8000 places in Europe and Asia into a conical projection. However, the approximately 5100 historical European place coordinates were only very inaccurately known.In addition, they were of different

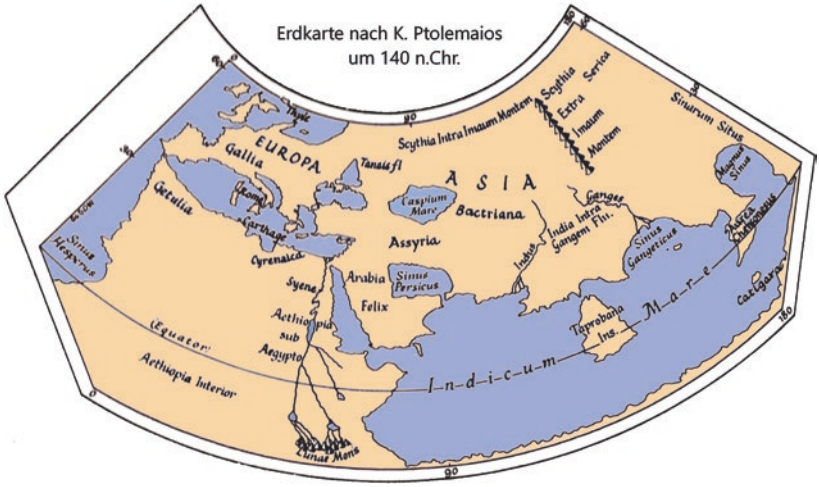

Fig. 18.12 Reconstruction of the world map according to Ptolemy (Wikimedia Commons), coloured by the author

accuracies, as they came from different sources. Figure 18.12 shows the representation of Eurasia and Africa.

The writing was still in the hands of Cassiodorus in 561/62, but the work did not become known in Europe until 1397, when a copy from Byzantium was translated from Latin in Florence. The most famous edition is by Sebastian Münster (1544), which was published under the name *Cosmographia*. Since original maps by Ptolemy have been lost over time, the *Cosmographia* has been added with new maps several times in the following period. A beautifully painted edition with 26 color maps was created around 1300 in Constantinople. It survived the conquest of Constantinople in 1453, but then fell into complete oblivion. It was not until 1927 that it was rediscovered by A. Deissmann in the holdings of the old library of the Topkapi Palace (Codex Seragliensis GI 57). The Codex latinus[5] V F.32 of the National Library of Naples with 27 color plates is also colorful.

Since the Greek edition of C.F.A. Nobbe from 1843/45 no newer edition has been available. Also in English there was only a partial translation of the theoretical chapters by J.L. Berggren[6] and A. Jones. Therefore A. Stückelberger founded a Ptolemaios research institute in Bern, which presented a bilingual, scientific edition[7] in 2006, three years later a supplementary volume followed.

[5] Pagani L. (ed.): Ptolemäus' Cosmographia – Das Weltbild der Antike, Parkland 1990.

[6] Berggren J.L., Jones A.: Ptolemy's Geography: An Annotated Translation of the Theoretical Chapters, Princeton University Press 2000.

[7] Stückelberger A., Graßhoff G. (Hrsg.): Klaudios Ptolemaios – Handbuch der Geographie, Schwabe Verlag, Basel 2006[1], 2017[2].

For the Ptolemaic places in Germania Magna, it has been possible for the scientists of TU Berlin to transfer the ancient coordinate data to the modern geographical coordinate system using some reference points (such as the mouth of river Weichsel and the city of Bonn). For some places, whose coordinates were transformable and could be historically assigned, it turned out that the historical longitudes were on the average too large by the factor 1.4. This can be explained by the use of two different lengths for a "stadium". Marinos from Tyros sets a stadium to 222.22 *m,* Eratosthenes but to 158.73 *m,* the ratio is exactly 7/5 (according to Kleineberg[8] et al.).

Through the work of the research group led by A. Kleineberg, in addition to towns, historical rivers, mouths and trade routes could be identified, including the famous Bernsteinstraße (trade route for amber). Coming from the Adriatic, the Bernsteinstraße reached Carnuntum/Petronell at the Danube, where it left the Roman Empire. On the northern side of the Danube, it led through Germania along the Marchtal to Parienna/Lundenburg and further to Brünn. Here it met a route that branches off in Scarbantia/Sopron, already in the Roman province of Pannonia Superior, passes the Danube at Vindobona/Vienna and runs through the Weinviertel (region of viticulture). From Eburodunum/Brünn it went in a northeasterly direction to the Mährische Pforte and from there to the mouth of the Vistula; Ptolemy did not know the further way to Samland. Also important was the trade route that led via Prague to the Elbe (access to the North Sea); a road also leads into this route from *Eburodunum*/Brünn.

The results of the research group Kleineberg were published in two books[9, 10].

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Nicomachus of Gerasa

19

Nikomachos (Νικὸμαχος) was a late Pythagorean from Gerasa[1] (today Jerash in Jordan), who lived around 60–120 AD and probably lived in Alexandria. Figure 19.1 shows Nikomachos to the right of Plato, who is holding the book "Musica". His life data can be determined quite accurately. On the one hand, he quotes the author Thrasyllos himself, who died in 36 AD; this sets a lower time limit. On the other hand, he is mentioned by Apuleius (125–179), which limits his age to the top. John M. Dillon[2] comes to a different dating, which sets the year of death to 196. The reason is curious: Proclus himself was convinced, according to his biographer, of being a reincarnation of Nicomachus. In general, the late Pythagoreans believed firmly in a rebirth period of 216 years. Since Proklos was born in 421 AD, this results in the given date of death. This dating is controversial; above all, it is very doubtful whether Proclus refered himself to this cycle.

He is not to be confused with the son of Aristotle, to whom Aristotle dedicated his Nicomachean Ethics. It is believed that he worked at the school of Alexandria at times. From the preface to his "Harmony Theory" we read that the book was written on behalf of a high-ranking lady on the journey *in the manner of the ancients;* i.e. in the manner of the traveling philosophers. The American researcher F.R. Levin[3] suspects that the lady was none other than Pompeia Plotina, the wife of the Roman emperor Trajan, who later deified his wife (Augusta).

The conquest of Alexandria in 31 BC ended the Ptolemaic Kingdom and made Egypt a Roman province. Nikomachos was such a famous mathematician that he

[1] Gerasa appears in the New Testament under various names: Matth.8,28; Markus 5,1; Lukas 8,26.

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Fig. 19.1 Nikomachos and Plato, manuscript Cambridge (eleventh century), Science Photo Library

was still honored by generations. Proklos (410–485 BC) saw him as a true soulmate, *Porphyrios*[4] from Tyros (233–301 BC) counted him among the most famous Pythagoreans in his Pythagoras biography; an (anonymous) early Christian pamphlet notes in *Altercatio Ecclesiae:*

> *Arithmeticam Samius Pythagoras invenit, Nicomachus scripsit* (Pythagoras of Samos invented arithmetic, Nikomachos wrote it down).

Isidor, Bishop of Sevilla, informs in his encyclopedia (III, 2):

> The first one to write about the sciences of numbers was, according to the Greeks, Pythagoras; then Nikomachus more accurately explained it. Apuleius and Boethius translated their writings.

His countryman Iamblichos of Chalcis (ca. 240–320 AD) wrote about him:

> We find that Nikomachos in his arithmetic treated everything about this subject in accordance with the doctrine of Pythagoras. For this man is a great scholar and had teachers who were very experienced in mathematics. Apart from that, he handed down the knowledge of arithmetic with great accuracy; he thereby shows a commendable regularity and a theory formation that offers a successful application of scientific principles.

The standard output of his works was carried out by Hoche[5]. Unlike many Pythagoreans, two of his works are completely preserved, the "Handbook of Harmony Theory" and the "Introduction to Arithmetic[6]". In his theory of harmony he presents the music theory of the Pythagoreans based on numerical proportions, going into the Pythagorean idea of spherical music. Here he tells the well-known

[4] Porphyry of Tyre: The Life of Pythagoras, pp. 123–136, in the anthology Guthrie.

[5] Nicomachi Geraseni Pythagorei introductionis arithmeticae libri II, Hoche R.(Hrsg.): Teubner, Leipzig 1866.

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anecdote that Plato, when passing by a forge, realized that the sound depended on the weight of the forge hammers. He treats the harmonies only from the point of view of their number-theoretical relevance; the musical practice does not interest him. In the work he emphasizes that he will write a complete work on music theory. It is possible that this is the book "On Music", of which Eutokios reports in his commentary on Archimedes. Part of this writing can be deduced from the work "De institutione musica" by Boethius, since he has drawn extensively from the *Harmony*.

Another work of Nicomachus '"Theology of Arithmetic" is lost; its content corresponds largely to the similarly named book of Iamblichus, which Nicomachus used as a source. Since the number ten was considered holy, the book contains the Pythagorean interpretation of the first ten numbers (decade). According to Aetius its motto is: "The nature of numbers is the decade." The content of *Theology* can be summarized by the following table:

1	Monad	Unit, principle of all things, essential, indivisible, point (in geometry)
2	Dyad	Principle of duality, first female (= even) number, line (geometrical)
3	Triad	Unit plus duality, first male (= odd) number, plane (geometrical)
4	Tetrade	First square, quadruplicity *Tetraktys* (elements, seasons, life stages), body (geometric)
5	Pentade	Marriage (= female + male), number of fingers, toes, Platonic solids, parallels of latitude on Earth
6		First perfect number, area of basic triangle (3; 4; 5)
7	Heptade	Number of heavenly bodies, number of first non-constructible polygon
8		First cubic number
9	Enneade	Second square
10	Decade	Set of all single-digit numbers, sum of the first tetrad (1 + 2 + 3 + 4), number of fingers or toes

19.1 From the *Arithmetica*

In the *Arithmetic* he deals mainly with number theory, but not for mathematical motives, but to provide the reader with the basics for understanding the philosophy of mathematics. The mathematical sciences included arithmetic, music, geometry and astronomy at that time. The latter science also provided the cosmological significance of the numbers for the world's creation. In the book (I, 6) he writes about it:

> All things in the world that are naturally and artfully arranged […] appear according to the number distinguishable and ordered, controlled by the foresight and reason that produced the universe. For the pattern was given, as in a plan, controlled by the number, already existing in the thoughts of the creator god, only recognizable by numbers and at the same time spiritual, yet always the true and eternal being, so that in relation to this plan all things are created: time, movement, heaven, stars, rotations of all kinds.

On the role of arithmetic is found (I, 4):

> The arithmetic itself existed in the thoughts of the ruling God before all other things, as a model of an ordering principle (λὸγος), on which the artist of the universe relies in planning and the original form of the order, produced by all beings for the completion of their true existence.

The concept of numbers of the Pythagoreans is not understood uniformly; Nikomachos quotes in (I, 7) about the following opinions: A (natural) number has 3 properties; it is:

a) a limited number of concrete things, e.g. 5 sheep
b) an accumulation of monads (units), in the sense of a set
c) a configuration that arises from the one by successive addition of the unit, comparable to the addition using abacus beads

Therefore, the Greek word ἀριθμὸς (arithmos) cannot be translated adequately with the word *number*, which in modern language usage has a completely different meaning.

In the first book of arithmetic, Nikomachos presents the Pythagorean theory of even and odd numbers. He particularly emphasizes the role of even numbers, which when multiplied by any number always result in an even product, in contrast to odd numbers, which only for odd factors again an odd product. He refines these definitions by even-even numbers; these are numbers that, after splitting off an even divisor, have further even divisors, so the powers of two 2^n. Odd-even numbers are of the form $2^n \cdot p$.

The name prime numbers is derived from the fact that the numbers without divisor (except for one) were called primary, the composite as secondary numbers. Prime numbers are always odd with Nikomachos. Odd-even numbers are of the form $2p$. The prime numbers are listed in a table, he does not count the number 2 among the prime numbers.

3	5	7	11	13	17	19	23	29	31	37	41	43	47

etc. Mentioned by name is the sieve of primes of *Eratosthenes,* which is explained verbally and in great detail. The starting point of his description is the list of all odd numbers from 3; even numbers do not appear. He writes in (I, 13):

> The method for generating these [primes] is called the *sieve* of Eratosthenes; for if one takes the odd numbers indiscriminately, they will be separated—like with an instrument or sieve. […] The nature of the sieve is as follows: I write all the odd numbers, starting with three, in a line as long as possible, and I start with the first and test it for divisors […].

Further, composite and perfect numbers are defined. Only proper divisors are considered, just as with Euclid; that is, no number counts as a divisor of itself. A

number is called *perfect,* if it is equal to the sum of its proper divisors. The perfect numbers form the boundary between the abundant and the deficient. A number is called *abundant,* if it is greater than the sum of its divisors. Nikomachos gives the number 12 as an example; 12 is abundant, since the sum of proper divisors gives: $1 + 2 + 3 + 4 + 6 = 16 > 12$. A number is *deficient,* if it is less than the sum of its proper divisors. He uses the number 14 as an example; 14 is deficient because of $1 + 2 + 7 = 10 < 14$.

Nikomachos still defines a variety of categories for numbers, the discussion of which would lead too far here and which has limited mathematical utility. His terms are *superparticular, subsuper-particular, sesquitertius* and similar; the number terms will later be used in the medieval number game "Rithmomachia" (after Folkerts[7]). All odd numbers < 1000 are deficient, except 945.

Nikomachos creates the perfect numbers tabularly; a proof like [Euklid IX, 36] he does not provide. First, he writes down all even-even numbers, that is, the two powers of numbers, in a row, starting with the one. Then he calculates the partial sum sequence of this, puts it in the second row and strikes out all non-prime numbers. In the third row, he multiplies the numbers standing above, which are not crossed out.

Since he does not fully calculate this table, he only finds the first four perfect numbers and wrongly concludes that the ones digits of these numbers are alternately 6 or 8. The correct statement is that the last digit is always 6 or 8, but the sequence is not alternating. This can be seen in the extended list of the first eight perfect numbers.

The algorithm of Euclid [VII, 2] for determining the greatest common divisor (gcd) in the subtractive form is simplified by Nikomachos by generally subtracting the smaller number from the larger. The calculation of the gcd according to Nikomachos therefore runs recursively according to

$$\gcd(a,b) = \begin{cases} \gcd(a-b,b); & a > b \\ \gcd(b-a,a); & a < b \end{cases}$$

1	2	4	8	16	32	64	128	256	512	1024	2048	4096
~~1~~	3	7	~~15~~	31	~~63~~	127	~~255~~	~~511~~	~~1023~~	~~2047~~	~~4075~~	8191
	6	28		496		8128						s. unten

1	$2^1(2^2-1) = 6$
2	$2^2(2^3-1) = 28$
3	$2^4(2^5-1) = 496$
4	$2^6(2^7-1) = 8128$
5	$2^{12}(2^{13}-1) = 33\,550\,336$

[7] Folkerts M.: Rithmomachia, a Mathematical Game from the Middle Ages, in the anthology Folkerts, pp. I–XXIII.

6	$2^{16}(2^{17}-1) = 8\,589\,869\,056$
7	$2^{18}(2^{19}-1) = 137.438.691.328$
8	$2^{30}(2^{31}-1) = 2\,305\,843\,008\,139\,952\,128$

The number example GCD (21, 49) he determines as follows

$$21 < 49 \Rightarrow \gcd(21,49) = \gcd(21,28)$$

$$21 < 28 \Rightarrow \gcd(21,28) = \gcd(21,7)$$

$$21 > 7 \Rightarrow \gcd(21,7) = \gcd(14,7)$$

$$14 > 7 \Rightarrow \gcd(14,7) = \gcd(7,7)$$

Since Nikomachos does not know the zero, he cannot carry out the last subtraction here. He therefore writes that the end is reached when the same number results. So it applies here $gcd(21,49) = 7$.

In the book (II, 12) the figured numbers appear again in tabular form, with he pointing to the same differences in each *column*.

Triangular numbers	1	3	6	10	15	21	28	36
Square numbers	1	4	9	16	25	36	49	64
Pentagonal numbers	1	5	12	22	35	51	70	92
Hexagonal numbers	1	6	15	28	45	66	91	120
Heptagonal numbers	1	7	18	34	55	81	112	148

He also discovers the following law of formation: Every polygonal number is the sum of the number above it in the same column and the triangular number from the column before. The eighth hexagonal number 120 is thus the sum of the eighth pentagonal number 92 and the seventh triangular number 28. He also observes that every square number is the sum of two triangular numbers:

$$n^2 = \frac{1}{2}(n-1)n + \frac{1}{2}n(n+1)$$

Spatial figurate numbers, such as pyramidal and spherical numbers, are also discussed in detail.

19.2 Proportions and Averages

Averages are also mentioned by Plato. In the Timaeus (32B) he needs two averages to unite the four elements:

> … so that fire should behave to air as air does to water, and as air does to water, so water does to earth; and so [God] bound and put together the universe.

Plato had previously described a geometric mean in Timaeus 32A:

> For if, of three numbers or measures or forces of any kind, the middle one bears the same ratio to the last as the first to it, and again the last bears the same ratio to the first as the middle to it…

In sections (II, 21) to (II, 29), Nicomachus deals with proportions and the associated averages. The arithmetic proportion (II, 23) is explained using the sequence of natural numbers. Four numbers $(a; b; c; d)$ are in arithmetic proportion if

$$a - b = c - d$$

Here the sequence of differences is constant; each number is the arithmetic mean of its two neighbors

$$n = \frac{(n-1) + (n+1)}{2}$$

The geometric proportion (II, 24) is demonstrated using powers of two and three. Four numbers $(a; b; c; d)$ are in geometric proportion if

$$a : b = c : d$$

Here the sequence of quotients is constant; each number is the geometric mean of its two neighbors

$$2^n = \sqrt{2^{n-1} \cdot 2^{n+1}}$$

The harmonic proportion (II, 25) of three numbers $(a; b; c)$ is defined by

$$(a - b) : (b - c) = a : c$$

If three strings of lengths $\left(1; \frac{2}{3}; \frac{1}{2}\right)$ are struck, they are in harmonic proportion because of

$$\left(1 - \frac{2}{3}\right) : \left(\frac{2}{3} - \frac{1}{2}\right) = 1 : \frac{1}{2}$$

It is not recognized that in the harmonic series each member is the harmonic mean $H = \frac{2ab}{a+b}$ of its two neighbors

$$\frac{1}{n} = \frac{2 \cdot \frac{1}{n-1} \cdot \frac{1}{n+1}}{\frac{1}{n-1} + \frac{1}{n+1}}$$

The inequality between the three mean values is known to him. If b is the mean of a and c, then

$$a : b = b : c \quad \text{geom. mean}$$

$$a : b < b : c \quad \text{arith. mean}$$

$$a : b > b : c \quad \text{harm. mean}$$

These three mean values already occur in Eudoxus. The highlight of section II, 19 is the appreciation as the most perfect of all proportions

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b$$

His commentator Iamblichos calls this proportion *harmonia* and writes:

> There is still the most perfect proportion of all to describe; it is three-dimensional and comprehensive and extremely useful for all progress that can be made in music and natural sciences …. They say that this [proportion] is a discovery of the Babylonians and came to Greece through the mediation of Pythagoras. Many Pythagoreans have used it, like Aristeos of Kroton, Timaios of Locris, Philolaos, Archytas of Tarent and many others. Plato also mentions it in Timaios [36A].

19.3 The Nikomachos Theorem

In arithmetic (II, 20) Nikomachos' own discovery can be found. He explains this with the following table

$$1 = 1 = 1^3$$

$$3 + 5 = 8 = 2^3$$

$$7 + 9 + 11 = 27 = 3^3$$

$$13 + 15 + 17 + 19 = 64 = 4^3$$

These formulas are a special case of the following series

$$\sum_{i=1}^{n} [n(n-1) - 1 + 2i] = n^3 \qquad (*)$$

By successive addition of odd numbers, all third powers (cubes) can be generated. The addition of the above equations gives

$$1^3 + 2^3 + 3^3 + 4^3 = 1 + 3 + 5 + \cdots + 19 = 100 = 10^2$$

In general, the *Theorem* of Nikomachos can be proven

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

Figure 19.2 shows a geometric illustration of Ned Gulley's theorem[8]. With summation formulas it can be written

[8] Gulley N., Shure L. (eds.): Nicomachus's Theorem, Matlib Central, March 4 2010.

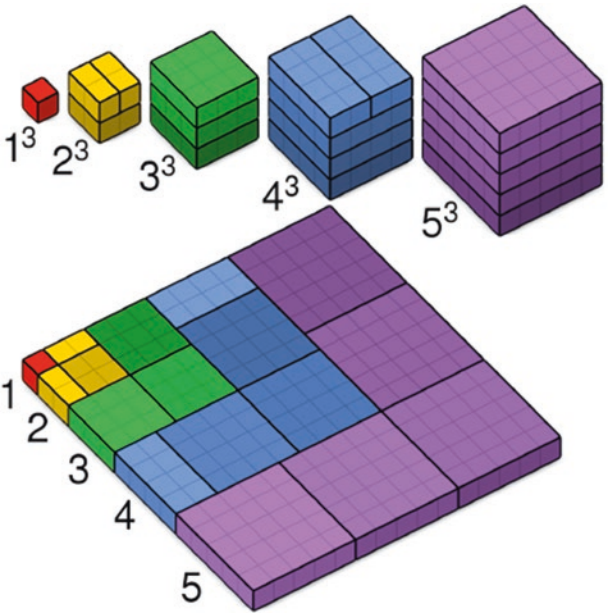

Fig. 19.2 Geometric interpretation of Nicomachus's theorem (Wikimedia Commons)

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left(\frac{n^2+n}{2}\right)^2$$

A proof can be found in Aryabhata's work Aryabhatiya (499). According to (*)

$$n^3 = \left(n^2-n+1\right) + \left(n^2-n+3\right) + \left(n^2-n+5\right) + \cdots + \underbrace{\left[n^2-n+(2n-1)\right]}_{n^2+n-1}$$

Adding up shows

$$\sum_{k=1}^{n} k^3 = \sum_{1\le k\le\left(n^2+n-1\right)}^{n} k = \sum_{k=1}^{\frac{1}{2}n(n+1)} (2k-1) = \left(\frac{n^2+n}{2}\right)^2$$

To reach the holy number 10, he invents seven more means in section (II, 28)—in addition to the mean values mentioned above—which are later also mentioned by Pappos. They are defined by the following proportions with $a > b > c$

$$(b-c) : (a-b) = a : c \quad (4)$$

$$(b-c) : (a-b) = b : c \quad (5)$$

$$(b-c) : (a-b) = a : b \quad (6)$$

$$(a-c) : (a-b) = a : c \quad (7)$$

$$(a-c):(a-b)=a:c \quad (8)$$

$$(a-c):(a-b)=b:c \quad (9)$$

$$(a-c):(a-b)=b:c \quad (10)$$

The so-defined means will not be used later. Curiously, mean (6) yields the following "crooked" term after transformation

$$b=\frac{c-a+\sqrt{c^2-2ac+5a^2}}{2}$$

For the unit interval [0; 1], that is, for $a=1, c=0$, the Golden Ratio φ

$$\varphi=\frac{-1+\sqrt{5}}{2}$$

results here.

He considers the binomial formula found there to be valuable, which is known as *regula nicomachi* in the Middle Ags.:

$$\left(\frac{a+b}{2}\right)^2=ab+\left(\frac{a-b}{2}\right)^2$$

This allows squares to be calculated, like

$97^2=(\frac{100+94}{2})^2=100\cdot 94+(\frac{100-94}{2})^2=9409.$

Nikomachos' arithmetic is the oldest book of antiquity dedicated specifically to arithmetic. Although there is still the arithmetic of Diophantus; but this work was unknown throughout the Middle Ages. Nikomachos' work had a similar importance for arithmetic in the following centuries as Euclid's *Elements* had for geometry. For it was translated into Latin by Boethius in his work *De institutione arithmetica* and thus became popular in monastic schools and universities throughout the Middle Ages.

19.4 From the Commentary of Lamblichos

Iamblichos of Chalcis (ca. 240-325 AD) was an ancient Greek philosopher of the Neoplatonic school; his place of birth was Chalcis ad Belum (today Qinnesrin) belonged at that time to the province Syria Coele. His first teacher was called Anatolios, a Neoplatonist; later Iamblichos joined the only a few years older Neoplatonist Porphyry, who was a student of Plotinus and lived in Rome. However, Iamblichos did not take over his teaching and founded his own school in Apameia on the Orontes. In addition to philosophy, he also devoted himself to

mathematics, his commentary[9] on Nicomachos provides valuable hints (see references). The teachings of Nikomacos influenced in particular Syrianos (ca. 375-437 AD) at the new academy in Athens.

A special number-theoretical achievement of a Pythagorean has Nesselmann[10] discovered in the Nicomachus commentary of Iamblichus . In painstaking translation, Nesselmann could read from the text that Thymaridas of Paros has treated the following special linear equation system (here in modern notation):

$$x_1 + x_2 + x_3 + \cdots + x_n = s(n \geq 3)$$

$$x_1 + x_2 = a_1$$

$$x_1 + x_3 = a_2$$

$$x_1 + x_4 = a_3$$

$$\cdots \quad \cdots \quad \cdots$$

$$x_1 + x_n = a_{n-1}$$

Thymaridas is one of the 218 Pythagoreans named by Iamblichus. Iamblichus (XXXIII, 239) reports in his biography of Pythagoras[11] that Thymaridas suddenly lost his wealth. Therefore, the Pythagorean Thestor from Posidonia expended considerable resources as a gesture of friendship to buy back his former property.

The solution method verbally formulated by Thymaridas is in modern writing:

$$x_1 = \frac{a_1 + a_2 + a_3 + \cdots + a_{n-1} - s}{n-2}$$

$$x_2 = a_1 - x_1$$

$$x_3 = a_2 - x_1$$

$$\cdots \quad \cdots \quad \cdots$$

$$x_n = a_{n-1} - x_1$$

Proof: The summation of the equations $x_1 + x_{i+1} = a_i$ yields

$$(n-1)x_1 + \underbrace{(x_2 + x_3 + \cdots + x_n)}_{s - x_1} = (a_1 + a_2 + a_3 + \cdots + a_{n-1})$$

[9] Iamblichi in Nicomachi arithmeticam introductionem liber, Pistelli E., Ulrich Klein U. (Hrsg.), Teubner 1975.

[10] Nesselmann G.H.F.: Die Geometrie der Griechen, G. Reimer-Verlag Berlin 1846, S. 235.

[11] Iamblichus, edited by Albrecht M.: Pythagoras - Legend, Doctrine, Life, Artemis Zurich 1963.

Simplifying yields the claim

$$(n-2)x_1 = (a_1 + a_2 + a_3 + \cdots + a_{n-1}) - s$$

The solution method was given the name *flower of Thymaridas*.

Iamblichus gives **Example 1:**

To find four numbers: The first and second is twice as large as the third and fourth together. The first and third is three times as large as the second and fourth together. The first and fourth is four times as large as the two middle ones. The sum of all four numbers is equal to five times the two middle ones.

Given is the linear system:

$$a + b = 2(c + d)$$

$$a + c = 3(b + d)$$

$$a + d = 4(b + c)$$

$$a + b + c + d = 5(b + c)$$

Iamblichos determines the product of all multiples, here 120, and sets it as the sum of the numbers. He first multiplies this sum by the multiple 2, divides by 3 and sets $a + b = 80$. Similarly, he multiplies 120 by the multiple 3, divides by 4 and sets $a + c = 90$. Analogously, he multiplies 120 by the multiple 4, divides by 5 and sets $a + d = 96$. Iamblichos continues:

> Since the 4 numbers have now been found in their connection, but not yet separated individually, the *Epanthema* of *Thymaridas* shows us the way to their separation from each other. If we add the 3 sums, i.e. 80, 90, 96, and subtract the initially found sum 120 from the total sum 266, we get the remainder 146 and half of that 73. The sought numbers result from the differences of 80, 90, 96 with half of 73.

Iamblichos realizes that his solution is not unique; he continues:

> These are the smallest values in whole numbers that observe the given proportions; if one divides or multiplies them by an arbitrary number, the numbers thus obtained will also satisfy the task.

The modern solution contains a parameter, as the given equations are linearly dependent. If you set the variable a as a parameter, you get the solution:

$$b = \frac{7}{73}a \;\therefore\; c = \frac{17}{73}a \;\therefore\; d = \frac{23}{73}a$$

The historical solution results in the parameter value $a = 73$.

As **Example 2** without previous transformations, the task (XIV, 49) from the *Anthologia Graeca* should be solved:

A crown made of gold, copper, tin and iron should be made with a total weight of 60 mines. The gold and copper share should be $\frac{2}{3}$, the gold and tin share should be $\frac{3}{4}$ and the gold and iron share should be $\frac{3}{5}$. What share do the individual metals have?

The equation system results in

$$x_1 + x_2 + x_3 + x_4 = 60$$

$$x_1 + x_2 = 40$$

$$x_1 + x_3 = 45$$

$$x_1 + x_4 = 36$$

With the given solution formula, successively follows

$$x_1 = \frac{1}{2}(40 + 45 + 36 - 60) = \frac{61}{2}$$

$$x_2 = 40 - \frac{61}{2} = \frac{19}{2}$$

$$x_3 = 45 - \frac{61}{2} = \frac{29}{2}$$

$$x_4 = 36 - \frac{61}{2} = \frac{11}{2}$$

The weight shares of gold, copper, tin and iron are $30\frac{1}{2}$, $9\frac{1}{2}$, $14\frac{1}{2}$ and $5\frac{1}{2}$ mines.

In his commentary on Nicomachus, Iamblichus mentions another method for generating square numbers: If the square n^2 is sought, one begins with the number one and counts up to n, then turns around, like a horse that returns after the race, and counts backwards to one (Fig. 19.3).

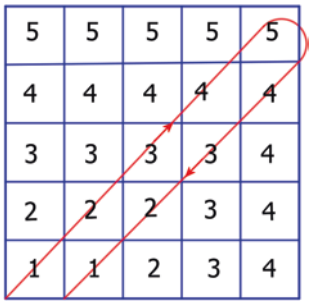

Fig. 19.3 Calculation of squares by means of sums of natural numbers

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix} 5\} \Rightarrow \text{sum} = 5^2$$

Finally, Iamblichus mentions the following rule: If you add 3 consecutive numbers, where the largest is divisible by 3, the repeated sum of digits (QS) will be 6. He gives the example:

$$997 + 998 + 999 = 2994 \Rightarrow \text{QS}(2994) = 24 \Rightarrow \text{QS}(24) = 6$$

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Theon of Smyrna

20

The life data of Theon (Θέων ὁ Σμυρναῖος) are only approximately known. The Suda calls him *Theon the Philosopher.* He comes from Smyrna, today's Ismir (Turkey). K. Ptolemaios attributes four Mercury- or Venus-observations to him in the years 127, 129, 130 and 132 during the reign of Emperor Trajan. He is also mentioned by name in the writings of Theon of Alexandria the *elder Theon*.

His Platon-Kommentar[1] *Expositio rerum mathematicarum ad legendum Platonem utilium* (Ed. R. Hiller, 1878) has been completely preserved. In the preface to his book, Theon proudly writes:

> Everyone will agree that the mathematical arguments used by Plato cannot be understood if one is not skilled in this discipline, and that the study of these things is neither unintelligent nor unprofitable in other respects, as Plato himself would appear. The one who is trained in the whole of geometry, in the whole of music and astronomy will be happy to learn of Plato's writings; but this does not happen simply or quickly, because it requires a great deal of practice from youth. In order to grant all those who have missed this study but still want to acquire knowledge of these writings, this wishing should not be in vain; therefore, I have prepared a summary and brief description of the mathematical doctrines that are especially necessary for readers of Plato who want to not only experience arithmetic and music, and geometry, but also their applications in stereometry and astronomy, because without these studies, as he [Plato] says, it is not possible to achieve the best life, and in many ways he makes it clear that mathematics should not be ignored.

M. Cantor (p. 154) attributes to him a source that was not used by other Late Pythagoreans:

> What Theon of Smyrna therefore highlights as Pythagorean mathematical teachings must have been drawn from other, non-mythical writings, of which Porphyry and Iamblichus made no use in their biographies of Pythagoras. […] But these can […] only be

[1] Theon of Smyrna: Mathematics Useful for Understanding Plato, Lawlor R. (Hrsg.), Wizards Bookshelf 1979.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_20

> knowledge that preceded the Platonic writings according to Theon's best knowledge, and could be used in them.

The book consists of three parts: arithmetic (I), numerical laws of music (II) and astronomy (III). The mathematical part contains the usual definitions of even, odd (I, 5), prime and composite number (I, 6-I, 7), as they can also be found with Nikomachos. His definition of parallelogram numbers (I, 14), which are numbers with two factors, where the larger factor must be at least two times larger than the smaller one, is not used. In (I, 15) the square numbers are generated by successive addition of odd numbers. Rectangular numbers (heteromekes) can be found in (I, 17); they are considered differently than the triangular numbers (I, 19). Figured numbers are treated in (I, 18–30). Perfect, deficient and abundant numbers close the first part (I, 32).

The section on music (II, 38) also mentions the tetraktys ($1 + 2 + 3 + 4 = 10$), and there is also a description of Plato's lambda (see Chap. 7). The sections on means (II, 55–60) conclude the book's musical part. Six means are listed.

The astronomical part III quotes Eratosthenes and Archimedes and, using the first's value for the Earth's radius, calculates the Earth's volume according to the second's formula. The representation of the frequency ratios of the planetary orbits, the so-called music of the spheres, goes beyond the scope of the book.

According to T. Heath (II, p. 70), there are two mathematical topics that are found exclusively in Theon. The first is the result of the side-/diagonal numbers in section (I, 31), the second is the number-theoretical proposition that a square can never have the form $3n + 2$, $4n + 2$ or $4n + 3$ (section I, 20).

20.1 The Side or Diagonal Numbers

The question that arose in the context of the doubling of the square was whether there is a square number that is twice another. According to Proclus, such a number does not exist. However, it is possible to approximately specify such a number pair, e.g. (7; 50). It holds $50 = 2 \cdot 5^2 = 7^2 + 1$. Intuitively seen, this means that the area of a square with side 7 is almost twice as large as the square with side 5.

This property is mentioned by Plato *(Politeia 546)* in connection with the *marriage number* in a way that is difficult to understand. He writes

> The smallest ratio of those two human and divine numbers is 3 : 4; this, when connected with 5, yields two proportional numbers, after being tripled: The one, the same, taken the same number of times, namely 100 multiplied by itself; the other, however, has the same length as the former, but is oblong, consisting first of the 100-fold square number of one of the diagonals of a square, the side of which is 5, the diagonal of which is rational when 1 is subtracted, but irrational when 2 is subtracted, thus both becoming irrational-further consisting of the 100-fold cube of 3.

Here it is a matter of a twofold representation of the number 4800. On the one hand, as a 100-fold square of the (rational) diagonal to the side 5, each square

reduced by 1: $100\left(7^2-1\right)=4800$; on the other hand as a 100-fold square of the (irrational) diagonal to the side 5, each square reduced by 2:

$$100\left[\left(5\sqrt{2}\right)^2-2\right]=4800$$

Without ancient comments, the above Plato passage is difficult to understand. To explain this, Proclus in his *Politeia*-Commentary on the so-called side/diagonal numbers:"

> The Pythagoreans showed with numbers that the (squares of the) rational diagonals, which are next to the irrational ones, are one unit larger or smaller than the double (squares of the sides belonging to them). For since the unit forms the basis of things in every respect, it is clear—they say—that it can be both side and diagonal.
>
> Now let there be two units, one as the unit of the side, the other as the unit of the diagonal. In this case, the square of the rational diagonal is one unit smaller than the double (square of the side). Now add the diagonal unit (1 + 1 = 2) to the side unit and 2 side units (1 + 2 = 3) to the diagonal unit. In this way, the (new) square side is 2 and the (new) diagonal 3. The squares of these numbers are 4 and 9, the latter being one unit larger than the double square of the side ($9=2\cdot 2^2+1$). Add the corresponding diagonal 3 again to the square side 2 and twice the corresponding side 2 to the diagonal 3, so the side ($5=3+2$) or the diagonal 7 ($7=3+2\cdot 2$). The squares of these numbers are 25 and 49, the latter number being one unit smaller than the double square of the side 5: $49=50-1=2\cdot 5^2-1$. Therefore, Plato now says that the number 48 is 1 smaller than the square of the rational diagonal to the side 5, and 2 smaller than the square of the non-rational diagonal 50.

Proklos emphasizes here that, in all squares thus formed, the square of the rational diagonal d always differs by 1 from the double square of the corresponding side a

$$2a^2-d^2=\pm 1$$

The stepwise formation of side/diagonal numbers described by Proklos is already found in Theon's aforementioned main work.

> Let there be two units given, of which we choose one as the side, the other as the diagonal of a square. Now add the diagonal to the side and twice the side to the diagonal. Again, add the diagonal to the side and twice the side to the diagonal, then the side is now 5 and the diagonal 7. If we continue this addition in this way, it results alternately that the square over the diagonal always differs from the double square over the side by ± 1.

If we write the side/diagonal numbers in the n-th step as a_n and d_n, then the following formulas correspond to Theon's specifications

$$a_{n+1}=a_n+d_n$$

$$d_{n+1}=2a_n+d_n$$

The starting values of the recurrence are $a_0=d_0=1$. The following sections provide an interpretation beyond Theon in modern terms.

20.2 Geometric Interpretation

A geometric interpretation of this double sequence offers the figure of the alternate removal inside the square (Fig. 20.1). This figure arises when searching for a common measure of the diagonal and side of the square. In doing so, the side a_1 and diagonal d_1 of the square are reduced to the corresponding lengths a_0, d_0 of another (naturally similar) square.

The square $ADHK$ with the side a_1 and the diagonal d_1 is considered. The square $BCDE$ with the side a_0 and the diagonal d_0 is sought. The diagonal $|HD| = d_1$ is divided into the sub-distances $|HE| = a_1$ and $|ED| = a_0$. Producing the side BC provides the intersection point F with the side AH. The triangle $\triangle FGH$ is right-angled isosceles with the leg $|FG| = a_0$, thus $|FH| = d_0$ applies. Thus, $\triangle ABF$ is also right-angled isosceles to the leg $|AF| = a_0$ and the base $|FB| = d_0$. In total, therefore,

$$a_1 = a_0 + d_0$$

$$d_1 = a_0 + a_1 = 2a_0 + d_0$$

This is exactly Theon's recursion. To be shown is the sign change of the term $d^2 - 2a^2 = \pm 1$. Inserting the recursion formula shows

$$\begin{aligned} & d_n^2 - 2a_n^2 = (2a_{n-1} + d_{n-1})^2 - 2(a_{n-1} + d_{n-1})^2 \\ \Rightarrow\ & d_n^2 - 2a_n^2 = +\left(2a_{n-1}^2 - d_{n-1}^2\right) \\ \Rightarrow\ & d_n^2 - 2a_n^2 = -\left(d_{n-1}^2 - 2a_{n-1}^2\right) \end{aligned}$$

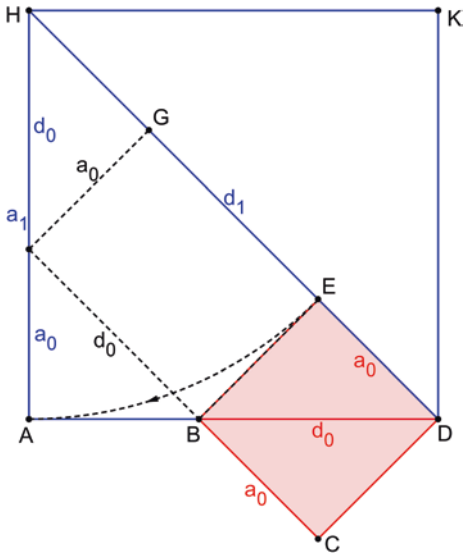

Fig. 20.1 Geometric interpretation of the diagonal, side numbers

The sign change continues until the beginning because of $a_0 = d_0 = 1$

$$\Rightarrow d_0^2 - 2a_0^2 = -1$$

The expression $y^2 - 2x^2 = \pm 1$ can also be considered a Diophantine equation; it is a special case of a so-called Pell equation, the naming was mistakenly done by Euler.

$$y^2 - dx^2 = 1 \quad \left(\sqrt{d} \notin \mathbb{Z}\right)$$

If a, b is a solution, then the terms $a' = a + 2b, b' = a + b$ also fulfill this equation

$$a'^2 - 2b'^2 = (a + 2b)^2 - 2(a + b)^2 = -a^2 + 2b^2 = -1$$

Since the initial condition $a = b = 1$ is also fulfilled here, there are thus infinitely many (integer) solutions.
If you set some numbers, you get the double sequence

$$a_2 = 2; d_2 = 3$$

$$a_3 = 5; d_3 = 7$$

$$a_4 = 12; d_4 = 17$$

$$a_5 = 29; d_5 = 41$$

$$a_6 = 70; d_6 = 99$$

The approximation $\frac{d_3}{a_3} = \frac{7}{5} \approx \sqrt{2}$ was already known to Aristarchus; the value $\frac{d_4}{a_4} = \frac{17}{12} \approx \sqrt{2}$ can be found in Heron. It is therefore to be expected that the sequence $\frac{d_n}{a_n}$ delivers approximation fractions for $\sqrt{2}$. A. Szabò attributes this discovery to the author E. Stamatis (1953). The transformation

$$y^2 - 2x^2 = \pm 1 \Rightarrow \frac{y}{x} = \sqrt{2 \pm \frac{1}{x^2}}$$

already comes from P. Fermat.

Here you can see the convergence for increasing x values: $\frac{y}{x} \to \sqrt{2}$. As above, it applies

$$x_{n+1} = x_n + y_n$$

$$y_{n+1} = y_n + 2x_n$$

Then, for the quotient, follows

$$\frac{y_{n+1}}{x_{n+1}} = \frac{y_n + 2x_n}{x_n + y_n} = \frac{\frac{y_n}{x_n} + 2}{1 + \frac{y_n}{x_n}}$$

Fig. 20.2 Table of the Double episode of Theon

x	y	y/x
1	1	1,000000000000
2	3	1,500000000000
5	7	1,400000000000
12	17	1,416666666667
29	41	1,413793103448
70	99	1,414285714286
169	239	1,414201183432
408	577	1,414215686275
985	1393	1,414213197970
2378	3363	1,414213624895
5741	8119	1,414213551646
13860	19601	1,414213564214
33461	47321	1,414213562057
80782	114243	1,414213562427

If one assumes the convergence against the value z, one obtains in total:

$$\lim_{x\to\infty} \frac{y_{n+1}}{x_{n+1}} = \frac{y_n}{x_n} = z \Rightarrow z = \frac{z+2}{1+z} \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$$

This gives the approximation fractions from the table below (Fig. 20.2). For the convergence, the theory of continued fractions results in:

$$\left|\frac{y}{x} - \sqrt{2}\right| \leq \frac{1}{2x^2}$$

Generalization

S. Giberson and T. Osler generalized Theon-Doppelfolge[2] so that it also provides an approximation for other roots $\sqrt{c}$ $(c \geq 3)$

$$x_{n+1} = x_n + y_n$$

$$y_{n+1} = y_n + (c-1)x_n$$

For $\sqrt{3}$ one obtains the sequence of approximation fractions here

$$\frac{1}{1}; \frac{2}{1}; \frac{10}{6}; \frac{28}{16}; \frac{76}{44}; \frac{208}{120}; \frac{568}{328}$$

[2] Shaun Giberson et Thomas J. Osler: Extending Theon's Ladder to any Square Root (www.rowan.edu/open/depts/math/osler/GibersonOsler.pdf) [01.05.2013].

20.3 Number Theory

The following sentence can be found in his writing on number theory: If m^2 is a (integer) square, then

(a) m^2 or $m^2 - 1$ is divisible by 3
(b) m^2 or $m^2 - 1$ is divisible by 4.

He writes more precisely: If m^2 is a square, then the following pairs of terms are whole numbers; one of the four cases applies

$$\frac{m^2-1}{3}, \frac{m^2}{4} \epsilon \mathbb{Z} \tag{20.1}$$

$$\frac{m^2-1}{4}, \frac{m^2}{3} \epsilon \mathbb{Z} \tag{20.2}$$

$$\frac{m^2}{3}, \frac{m^2}{4} \epsilon \mathbb{Z} \tag{20.3}$$

$$\frac{m^2-1}{3}, \frac{m^2-1}{4} \epsilon \mathbb{Z} \tag{20.4}$$

An example for case Eq. (20.1) is $m^2 = 4$, for case Eq. (20.2) $m^2 = 9$, for case Eq. (20.3) $m^2 = 36$ and finally for case Eq. (20.4) $m^2 = 25$. A modern proof is:

Case Eq. (20.3): $m = 6k \Rightarrow m^2 = 36k^2 \Rightarrow \frac{m^2}{3} = 12k^2 \Rightarrow \frac{m^2}{4} = 9k^2$

Case Eq. (20.4):
$m = 6k \pm 1 \Rightarrow m^2 - 1 = 36k^2 \pm 12k \Rightarrow \frac{m^2-1}{3} = 12k^2 \pm 4k \Rightarrow \frac{m^2-1}{4} = 9k^2 \pm 3k$

Case Eq. (20.1):
$m = 6k \pm 2 \Rightarrow m^2 = 36k^2 \pm 24k + 4 \Rightarrow \frac{m^2}{4} = 9k^2 \pm 6k + 1 \Rightarrow \frac{m^2-1}{3} = 12k^2 \pm 8k + 1$

Case Eq. (20.2):
$m = 6k \pm 3 \Rightarrow m^2 = 36k^2 \pm 36k + 9 \Rightarrow \frac{m^2}{3} = 12k^2 \pm 12k + 3 \Rightarrow \frac{m^2-1}{4} = 9k^2 \pm 9k + 2$

The proof becomes simpler by means of calculating *modulo*. Because of $36 = 4 \cdot 3^2$ follows

$$m = \pm 1 \mod 6 \Rightarrow m^2 = 1 \mod 36 \Rightarrow m^2 - 1 = 0 \mod 36 \Rightarrow m^2 - 1 = \begin{cases} 0 \mod 3 \\ 0 \mod 4 \end{cases}$$

On the topic of mean values Theon mentions that the number five is an arithmetic mean of numbers under 10; in the diagram this is represented so.

1 4 7
2 **5** 8
3 6 9

Could this be the first attempt of a magic square?

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Diophantus of Alexandria

21

The topic, on which I, the book, am struggling, friend, leads to the depths and repels through its harshness.
Whoever reaches my bottom by diving, … will undoubtedly win the main prize for geometry
and be considered a philosopher as well - Plato guarantees it with his work. (Epigram by Leon the Geometer on Diophantus' Arithmetica, Anthologia Graeca [IX, 578]. The last line is a reference to Plato's inscription Let no one ignorant of geometry enter.)

Similarly to Heron, the dating of Diophantus' life is disputed. Since he is neither mentioned by Nikomachos von Gerasa nor by Theon von Smyrna, he could have lived around 250 AD. Since he himself quotes Hypsikles von Alexandria (around 150 BC), he cannot have lived earlier. He is himself quoted by Theon von Alexandria (335–405 AD), so he must have lived before him. This provides a (unsatisfactory) limit of about 500 years.

His arithmetic is dedicated to a certain *Dionysus*. If this Dionysus is meant to be the Bishop of Alexandria (Bishopric of 248–264 AD), the above time frame would be confirmed. His lifetime could be set at around 200–284 AD. The dedication at the beginning of the *Arithmetica* to Dionysus reads:

> Since I know, venerable Dionysus, that you are eager to learn the solution to arithmetic problems, I have tried to explain the science of arithmetic to you … However, since the beginner only progresses slowly with the large number of numbers, and in addition easily forgets what has been learned, I have found it expedient to deal with those tasks which are suitable for a closer development and which explain the first elementary tasks in an appropriate manner, proceeding from the simplest to the more complicated. For in this way the beginner will be comprehensible, and the procedure will be imprinted in his memory, since the whole treatise comprises 13 books.

The Byzantine polymath M. Psellus (ca. 1017–1078) reports that a Diophantus and Anatolius dealt with Egyptian mathematics:

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_21

> The very learned Anatolius had gathered together the essential parts of the subject matter and dedicated his work to Diophantus.

This could indicate that Anatolios was a student of Diophantus. Since the former became bishop of Laodicea in 280, the lifetime of Diophantus seems to be confirmed around 250.

On the other hand, Jacob Klein found that a writing from the time of Emperor Nero exists about an astronomer Diophantus. If this assignment is correct, Diophantus would have to be set in the first century AD. A writing by Bar Hebraeus (around 1250), called al-Faraj by the Arabs, sets Diophantus in the time of the Roman Emperor Julian, the Apostate (the rebel) (reign 361–363). Diophantus would then be a younger contemporary of Theon, which hardly speaks for being quoted by Theon.

It is known of three works by Diophantus: The first is the *Arithmetica* in 13 books, of which six are preserved in Greek and four in Arabic, a treatise on *polygonal numbers* and a book on *porisms.* W. Knorr conjectures that only the first six books have been preserved in Greek, since Hypatia wrote a commentary on these precisely. This thesis is rejected by A. Cameron[1] . The work on porisms is lost, that on polygonal numbers is only partially preserved. Diophantus mentions three porisms in his *Arithmetica,* one of which states: "The difference of two arbitrary cubes is also the sum of two other cubes."

The representation of Diophantus in current literature is strangely ununified. Authors like N. Schappacher[2] or T. Heath deny him the ability to have independently composed such a variety of tasks; rather, he is said to have been the head of an author team that evaluated ancient Egyptian and Babylonian texts. Heath[3] explains:

> It is obvious that one person cannot be the author of all these problems from Book I-VI. There are even inequalities contained therein; some problems are below the level of the rest. … Furthermore, it seems likely that Problem (V, 30), which is in the form of an epigram, is from another author. The *Arithmetica* is undoubtedly a collection, just as Euclid's *Elements* are.

H. Hankel[4] writes completely off the wall:

> In the midst of this sad wasteland suddenly rises a man with youthful vigor: Diophant. … He lived in Alexandria; his place of birth is unknown; if conjecture were allowed, I would say he was not Greek; … if his writings were not written in Greek, no one would come to the conclusion that they had sprung from Greek culture.

---

[1] Cameron A.: Isidore of Miletus and Hypatia, On the Editing of Mathematical Texts, Greek, Roman and Byzantine Studies, 31 (1990), pp. 103–127.

[2] Schappacher N.: Wer war Diophant? Math. Semesterberichte, 45 (1998), S. 141 ff.

[3] Heath Th. (Hrsg.): Diophantus of Alexandria, Martino Publishing 2009, S. 124.

[4] Hankel H.: Zur Geschichte der Mathematik in Altertum und Mittelalter, Teubner Leipzig 1874, S. 157.

Burton David in his History of Math (1991, p.223) simplifies:*Diophantus was most likely a Hellenized Babylonian*. More positively, this is seen by M. Cantor[5] :

> For us, Diophantus with his name, which occurs several times in Greece, is a real Greek, a pupil of Greek science, even if such a one who towers far above his contemporaries is Greek in what he accomplishes, as in what he is unable to accomplish.

J. Klein[6] sees Diophantus as an inspiration and source for Vieta:

Modern algebra and modern formalism grew out of Vieta's occupation with Diophantus.

According to Isabella Bashmakova[7] Diophantus knows the sign rules of algebra. In problem (III, 8) he simplifies the equation $x^2 + 4x + 1 = 2x + 7$ by bringing it over to $x^2 + 2x - 6 = 0$; i.e., he has carried out the subtraction $1 - 7 = -6$. Similarly, in problem (VI, 14) he subtracts $\left(90 - 15x^2\right)$ from 54 with the result $15x^2 - 36$; here he applies the sign rule $-(-x) = x$. However, negative values are not accepted as solutions. In problem (V, 2) the equation $4 = 4x + 20$, results, which Diophantus calls *absurd*. A statement on the discussion of the geometric algebra has been included in Bashmakova's algebra book[8] at Bashmakova's special request.

In an extensive study, K. Barner[9] found 33 instances in Diophantus where negative numbers occurred. The Bourbaki collaborator A. Weil writes in his *History of Mathematics:*

> It happens more often (and this makes it interesting): The early appearance of concepts and methods that only later come to the attention of mathematicians. It is the task of the historian to free them [from their context] and to examine their influence or non-influence on later developments.

The popular author E.T. Bell said about him in his work *The last Problem [Fermat]:*

> Probably he [Diophantus] was a mathematical genius with new ideas. There were several of them.

I. Kleiner (S. 3) ackowledges Diophantus' work as algebra and attributes the following progress to him:

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[5] Cantor M.: Vorlesungen über die Geschichte der Mathematik, Teubner Leipzig 1907, S. 466.

[6] Klein J.: Greek Mathematical Thought and the Origin of Algebra, Dover 1992.

[7] Bashmakova I.: Diophant and diophantine equations, UTB Birkhäuser 1974, p. 38.

[8] Bashmakova I.: A new View of the Geometric Algebra of the Ancients, p. 163–176, in Bashmakova.

[9] Barner K.: Diophant and negative numbers. To two remarks Norbert Schappachers, Math. Schriften Kassel, Preprint 10/98.

He applies two fundamental rules of algebra, the transfer of a term to the other side and the cancellation of equal terms on both sides of an equation. Furthermore, he defines powers up to the 6th power of an unknown and their reciprocals. He calculates with negative terms and sets their product positive. He overcomes some previously valid regulations, such as that all terms are interpreted geometrically, that powers are limited to dimension 3 and that only quantities of the same dimension are added.

One of the few pieces of evidence we have about Diophantus is the famous puzzle in verse form from the *Anthologia Graeca* [XIV, 126] by an anonymous poet. It goes like this:

Under this mound lies Diophantus.
It is truly a wonder: The stone tells us,
cleverly, the man's age.
The god granted him youth for a sixth of his life,
then gave him downy beard for a twelfth,
stretched the wedding torch after another seventh,
said five years later graciously to the offspring.
The miserable boy: Only half as old as his father he became,
snatched away by frost, burned high on the woodpile!
Through arithmetic calculations the father sought four years long
to banish the pain before he himself died.

An inaccuracy can be found in line 8. One possible interpretation is that the son has half the current age of the father. Alternatively, the father's age at death may be meant here. If we set the father's age at death as the variable x, then in the second case we get the equation

$$\begin{aligned} x &= \tfrac{x}{6} + \tfrac{x}{12} + \tfrac{x}{7} + 5 + \tfrac{x}{2} + 4 \\ \Rightarrow x &= \tfrac{25}{28}x + 9 \Rightarrow x = 84 \end{aligned}$$

In the first case we find

$$\begin{aligned} x &= \tfrac{x}{6} + \tfrac{x}{12} + \tfrac{x}{7} + 5 + \tfrac{x-4}{2} + 4 \\ \Rightarrow x &= \tfrac{25}{28}x + 7 \Rightarrow x = 65\tfrac{1}{3} \end{aligned}$$

Most authors prefer the integer solution, since the task was obviously formulated so that 84 is the main denominator of all fractions.

Linear equations were solved in antiquity with the *method of false approach* (Regula Falsi). Here you put a supposed number into both sides of the equation and calculate the solution from the resulting difference. The first assumption is $x_1 = 42$. Inserting into the equation results in the difference

$$d_1 = x_1 - \frac{x_1}{6} - \frac{x_1}{12} - \frac{x_1}{7} - 5 - \frac{x_1}{2} - 4 = -4{,}5$$

The second assumption, $x_2 = 126$, shows the difference

$$d_2 = x_2 - \frac{x_2}{6} - \frac{x_2}{12} - \frac{x_2}{7} - 5 - \frac{x_2}{2} - 4 = 4{,}5$$

The Regula Falsi thus provides the desired solution

$$x = \frac{x_1 d_2 - x_2 d_1}{d_2 - d_1} = \frac{42 \cdot 4{,}5 - 126 \cdot (-4{,}5)}{4{,}5 - (-4{,}5)} = \frac{756}{9} = 84$$

The *Arithmetica* was translated into Arabic by the scholar Qusṭā ibn Lūqā who worked in Baghdad around 900 AD. Ibn Lūqā (820–912) was Syrian and traveled to Byzantium, where he brought back Greek writings. His translation influenced later Arab mathematicians such as al-Karajī. Parts of it must have been known in Constantinople in the thirteenth century, since M. Planudes wrote a commentary on the first two books. Surprisingly, F. Sezgin found ibn Lūqā's translation of four books of the *Arithmetica.* in an Iranian library in 1968.

In the West, it was not until the Renaissance that Regiomontanus discovered a Greek Diophantus exemplar in a Venetian library in 1463. It was not until the translation of C.-G. Bachet (de Meziriac) into Latin that mathematicians Fermat, Euler, Legendre and others were stimulated to produce their own works, which were significant for number theory. Figure 21.1 shows the title page of the French edition of 1621.

The task (V, 33) was written in the form of a poem and was considered as an insert by G. Nesselmann[10] and interpreted as follows:

> Two kinds of wine, 8 drachmas the measure, and worse to 5 only
> mixed the good lord his servants for the feast.
> What he paid as the price for both was a square number;
> if you add 60 to the square, you see, so you have a second square;
> now notice, the root shows you how much measure that one bought in total.
> And now tell me how much of the better wine,
> and how much of the to 5, were mixed together?

If they really came from Diophantus, they are interesting because they lead to a rarely occurring quadratic inequality. Diophantus sets x as the total amount, the total price is then $x^2 - 60 \to \square$. Then he tries to divide the total price into two sums so that one fifth of the one sum and one eighth of the other sum make up x. But this is only possible if it is true that

$$x > \frac{1}{8}\left(x^2 - 60\right) \wedge x < \frac{1}{5}\left(x^2 - 60\right)$$

The double inequality follows: $5x < \left(x^2 - 60\right) < 8x$. If 60 is added to the double inequality, it follows that

$$5x + 60 < x^2 < 8x + 60 \Rightarrow 11 < x < 12$$

If you set the total price equal to a square $x^2 - 60 = (x - z)^2$, you get $x = \frac{z^2+60}{2z} > 11 \Rightarrow z > 19$. Substituting the upper limit gives $x = \frac{z^2+60}{2z} < 12 \Rightarrow z < 21$. Thus, $z = 20$. Substitution shows $x = \frac{20^2+60}{40} = \frac{23}{2} \Rightarrow x^2 = \frac{529}{4}$. If you set the

[10] Nesselmann G., Die Algebra der Griechen, Berlin 1842, S. 395.

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V. C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Fig. 21.1 Title of the French Diophantos edition by Bachet (Wikimedia Commons)

individual quantities of the wine varieties equal to a, b, then the total price is $8a + 5b = x^2 - 60 = \frac{289}{4}$; the total quantity $a + b = x = \frac{23}{2}$. With $b = \frac{23}{2} - a$ it follows immediately $8a + 5\left(\frac{23}{2} - a\right) = \frac{289}{4}$ or $3a = \frac{59}{4}$. You thus get the solution

$$(a; b) = \left(\frac{59}{12}; \frac{79}{12}\right)$$

Curiosity on the side: From the wine prices of this task, S. Tannery drew the (unjustified) conclusion that Diophantus must have lived in the first century BC, since he estimated the wine prices in the second century to be much higher.

21.1 From Diophantus' Book I and II

Your soul to perdition, Diophantus, for the difficulty of your problems in general and of this one in particular![11]

Here are some typical tasks of Diophantus, which lead to quadratic terms, with solutions. Note that Diophantus only knows one unknown x; he can only introduce further unknowns as terms of x (usually linear).

Problem I, 17
To find four numbers, three of which form a sum of $\{22, 24, 27, 20\}$.

Solution: Let x be the sum of all four numbers; then for the four numbers according to the specification

$$x - 22; x - 24; x - 27; x - 20$$

Adding up gives the equation

$$4x - 93 = x \Rightarrow x = 31$$

The numbers sought are $\{9, 7, 4, 11\}$.

Task I, 24
To find three numbers, when a given fraction of the sum of two numbers is added, the results are all equal. The fraction for the first number is 1/3, for the second 1/4 and for the third 1/5. Take as the first number x and as the sum of the other two a number with a divisor 3, like 3.

The sum of all three numbers is therefore $x + 3$; the first plus 1/3 of the second and third number is $x + 1$; the second plus 1/4 of the first and third number is $x + 1$. Therefore, the triple of the second number plus the sum of all is equal to $4x + 4$, thus the second number is equal to $x + \frac{1}{3}$. Finally, the third number plus 1/5 of the first and second is equal to $x + 1$ or 4 times the third plus the sum of all is equal to $5x+5$, the third number is therefore $x + \frac{1}{2}$. Thus, the sum of all

$$x + (x + 1/3) + (x + 1/2) = x + 3 \Rightarrow x = \frac{13}{12}.$$

The numbers are $\left\{\frac{13}{12}; \frac{17}{12}; \frac{19}{12}\right\}$ or extended with common denominator $\{13; 17; 19\}$.

Task II, 8
To divide a given square number (16) into two squares.

Solution: The two squares are x^2 or $16 - x^2$. For the latter, Diophantus makes the approach $(2x - 4)^2$; the constant 4 is chosen that its square compensates for 16. This yields the equation

$$(2x - 4)^2 = 16 - x^2 \Rightarrow 5x^2 - 16x = 0 \Rightarrow x(5x - 16) = 0$$

[11] Byzantine scholion by J. Chortasmenos (1370–1437) on Diophantus (II, 8).

Solution is $x = \frac{16}{5} > 0$. The sought-after squares are therefore $\left\{\left(\frac{16}{5}\right)^2; \left(\frac{12}{5}\right)^2\right\}$.

Remark: When reading *this* task, P. de Fermat made the famous note in the margin of his Diophantus copy:

> He had found a wonderful proof that a decomposition into higher powers $z^p = x^p + y^p (p > 2)$ was not possible; only the edge was too small to grasp the proof.

This claim, called the *Great Theorem of Fermat*, was finally proved in 1995 by A. Wiles.

A general approach with the *slope m* and the constant z is $(mx - z)^2 = z^2 - x^2$. This yields

$$x = \frac{2mz}{m^2+1}; y = mx - z = \left(\frac{m^2-1}{m^2+1}\right)z$$

The sought-after squares are

$$x^2 = \frac{4m^2z^2}{\left(m^2+1\right)^2} \therefore y^2 = \frac{m^4 - 2m^2 + 1}{\left(m^2+1\right)^2}z^2$$

In particular, for $m = 3, z = 4$ the above solution $\left\{\left(\frac{16}{5}\right)^2; \left(\frac{12}{5}\right)^2\right\}$ results again.

A *modern* approach uses the representation as a circle $x^2 + y^2 = 16$.. You will immediately find a special point A of the circle at $(x_0; y_0) = (0; -4)$; a secant should be erected at this point. The straight line equation through A with unknown slope is $y = mx - 4$. The intersection shows

$$x^2 + (mx - 4)^2 = 16 \Rightarrow x = \frac{8m}{1+m^2}; x \neq 0$$

We choose $m = 2$, i.e. the secant $y = 2x - 4$. The intersection results in the point B $\left(\frac{16}{5}; \frac{12}{5}\right)$ (Fig. 21.2).

Computational solution: The circle mentioned above has the (rational) parameter representation (except for $(0; -4)$):

$$(x; y) = \left(\frac{8t}{1+t^2}; \frac{4\left(1-t^2\right)}{1+t^2}\right)$$

Substitution into the secant equation yields

$$\frac{4\left(1-t^2\right)}{1+t^2} = \frac{16t}{1+t^2} - 4 \Rightarrow t = \frac{1}{2} \Rightarrow (x; y) = \left(\frac{16}{5}; \frac{12}{5}\right)$$

Task II, 9

To divide a given number, which is the sum of two squares, into another square sum. The given number is $13 = 2^2 + 3^2$.

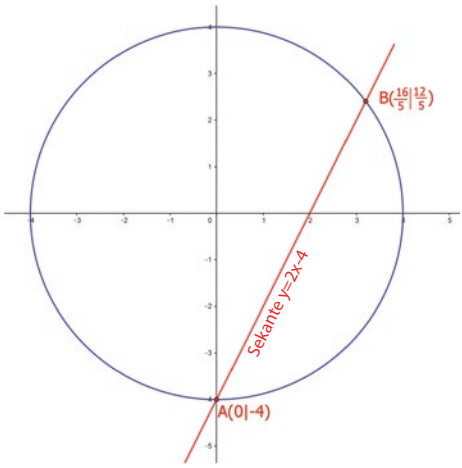

Fig. 21.2 For Diophantus II, 8

Solution: For the squares, Diophantus makes the ansatz $(x+2)^2$ or $(2x-3)^2$ so that the constants can be factored out. This results in the equation

$$13 = (x+2)^2 + (2x-3)^2 \Rightarrow x(5x-8) = 0$$

The solution $x = \frac{8}{5}$ provides the desired decomposition into squares $13 = \left(\frac{18}{5}\right)^2 + \left(\frac{1}{5}\right)^2$.

A general ansatz for the second square is $(mx-3)^2$. Setting it equal as above results in

$$\begin{aligned} 13 = (x+2)^2 + (mx-3)^2 &\Rightarrow \left(m^2+1\right)x^2 + (4-6m)x = 0 \\ &\Rightarrow x\left[\left(m^2+1\right)x + (4-6m)\right] = 0 \end{aligned}$$

This shows $x = \frac{|4-6m|}{m^2+1}$. The desired squares are

$$\frac{4\left(m^2+3m-1\right)^2}{\left(m^2+1\right)^2} \therefore \frac{\left(3m^2-4m-3\right)^2}{\left(m^2+1\right)^2}$$

For $m = 2$ in particular, the decomposition (as above) $13 = \left(\frac{18}{5}\right)^2 + \left(\frac{1}{5}\right)^2$ results. For $m = 3$ or $m = 7$, Diophantus does not find the decompositions

$$13 = \left(\frac{34}{10}\right)^2 + \left(\frac{12}{10}\right)^2 \therefore 13 = \left(\frac{69}{25}\right)^2 + \left(\frac{58}{25}\right)^2$$

Task II, 10
To find two square numbers having a given difference (60).

Solution: Diophantus makes the approach x^2 or $(x+3)^2$ for the two squares. Substitution gives

$$(x+3)^2 - x^2 = 60 \Rightarrow x = \frac{17}{2}$$

The two squares are therefore $\left(\frac{23}{2}\right)^2$; $\left(\frac{17}{2}\right)^2$.

A general approach for the second square is $(x+m)^2$. Equating as above gives

$$(x+m)^2 - x^2 = 60 \Rightarrow 2mx + m^2 = 60$$

This shows $x = \frac{60-m^2}{2m}$. The squares sought are therefore $\frac{(60-m^2)^2}{4m^2}$ and $\frac{(60+m^2)^2}{4m^2}$. This allows you to find other solutions, including two integers:

- $m = 2$ results in $60 = 16^2 - 14^2$
- $m = 6$ results in $60 = 8^2 - 2^2$

Task II, 11

For two given numbers (2; 3) a number is to be added, so that in each case a square (□) results.

Diophantus chooses the first number $x^2 - 2$, so that the sum with 2 is a square. The second sum is then $x^2 + 1$; this should be a square, he sets it equal to $(x-4)^2$. Diophantus writes next to it that the second number has the form $(x-a)$, where $x^2 > 2$ must be; he chooses $a = 4$. Thus the equation follows

$$x^2 + 1 = (x-4)^2 \Rightarrow x = \frac{15}{8}$$

The sought number is therefore $\left(\frac{15}{8}\right)^2 - 2 = \frac{97}{64}$. The squares result

$$\frac{97}{64} + 2 = \left(\frac{15}{8}\right)^2 \therefore \frac{97}{64} + 3 = \left(\frac{17}{8}\right)^2$$

Task II, 28

To find two squares such that their product added to each of the numbers is again a square.

Solution: We set the squares x^2, y^2; Diophantus knows only one unknown here. It must hold

$$\left.\begin{array}{l} x^2y^2 + x^2 = x^2\left(y^2+1\right) \\ x^2y^2 + y^2 = y^2\left(x^2+1\right) \end{array}\right\} \rightarrow \square$$

To make the first term a square, Diophantus sets $\left(x^2+1\right) = (x-2)^2$. This provides $x = \frac{3}{4}$. Analogously, it also follows: $\frac{9}{16}\left(y^2+1\right) \rightarrow \square$. Diophantus makes the approach

$$\frac{9}{16}\left(y^2+1\right) = \left(\frac{3}{4}y - 1\right)^2 \Rightarrow 9y^2 + 9 = (3y-4)^2$$

Solution is here $y = \frac{7}{24}$; the sought squares are thus $\left\{\left(\frac{3}{4}\right)^2; \left(\frac{7}{24}\right)^2\right\}$.

21.2 From Diophantus' Book III to V

Task III, 7
Three numbers are sought at equal intervals such that the sum of any pair gives a square.

Diophantus sets the three numbers without justification as $\{x; x+1; x-8\}$. The associated differences of the squares are

$$2x + 1 = 63 - 18x \Rightarrow x = \frac{31}{10}$$

The three squares result in $\left\{\left(\frac{31}{10}\right)^2; \left(\frac{41}{10}\right)^2; \left(\frac{49}{10}\right)^2\right\}$. An integer solution is obtained by expanding with 100: {961; 1681; 2401}. To find three numbers such the sums of pairs are the numbers just found. Half the sum gives 5043/2. Thus the sought numbers are {120.5; 840.5; 1560.5}.

Book IV offers some tasks in which equations of the third degree occur.

Task IV, 1
To divide a given number into two cubes, such that the sum of the bases is a given number. Diophantus sets the first number 370, the second 10.

Solution: So it is necessary to solve the following system

$$\begin{aligned} x + y &= 10 \\ x^3 + y^3 &= 370 \end{aligned}$$

As usual with Diophantus, he sets the summands equal to $(5 \pm x)$. To be fulfilled is

$$(5 + x)^3 + (5 - x)^3 = 370$$

Simplifying yields $250 + 30x^2 = 370$. The positive solution is $x = 2$; the sought bases are 7 and 3. The test confirms $7^3 + 3^3 = 370$.

Task IV, 15
To find three numbers such that the sum of any two numbers, multiplied by the third, results in a number from {35; 27; 32}.

Solution: If x, y, z are the three numbers, then it must hold

$$(x + y)z = 35 \therefore (x + z)y = 27 \therefore (y + z)x = 32$$

From the first equation follows $x + y = \frac{35}{z}$. Diophantus splits 35 into 10 and 25 as a test; this results in

$$x = \frac{25}{z} \therefore y = \frac{10}{z}$$

Inserted into the remaining equations, a *contradiction* results:

$$\frac{250}{z^2} + 10 = 27 \therefore \frac{250}{z^2} + 25 = 32$$

Therefore, 35 is split into 15 and 20; the equations result

$$\frac{300}{z^2} + 15 = 27 \therefore \frac{300}{z^2} + 20 = 32$$

These equations are identical to the solution $z = 5$. The sought numbers are $\{3; 4; 5\}$.

Task IV, 19

To find three numbers, the product of which in pairs, increased by one, is a square.

Solution: Since $x^2 + 2x + 1$ is a square, Diophantus sets the product of the first two numbers to $x(x + 2) = x^2 + 2x$. As a third number he chooses $(4x + 4)$. The product of the first and third number is then

$$x(4x + 4) = 4x^2 + 4x = (2x + 1)^2 - 1$$

To test, determine the product of the second and third number

$$(x + 2)(4x + 4) = 4x^2 + 12x + 8 = (2x + 3)^2 - 1$$

The desired indeterminate solution triple is

$$(x; x + 2; 4x + 4)$$

Task IV, 24

Divide a number into two summands so that their product results in a cubic number, decreased by its base. The number is 6.

Solution: The two summands are x and $6 - x$. For the corresponding product, the following should apply

$$x(6 - x) = y^3 - y$$

For y, Diophantus makes the linear approach

$$y = ax - 1 \tag{21.1}$$

First, he sets $a = 2$. Inserting results in

$$x(6 - x) = (2x - 1)^3 - (2x - 1)$$

Calculating results in the equation of the third degree

$$6x - x^2 = 8x^3 - 12x^2 + 4x$$

This equation has no rational solution except for zero. Therefore, Diophantus sets the value $a = 3 \Rightarrow y = 3x - 1$. Inserting results in

$$6x - x^2 = 27x^3 - 27x^2 + 6x$$

As a result of the elimination of the linear terms, a rational solution results

$$0 = 27x^3 - 26x^2 = x^2(27x - 26)$$

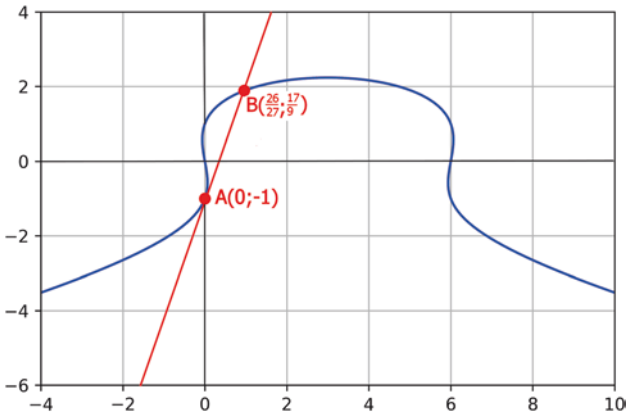

Fig. 21.3 Elliptical curve (with interchanged axes) and tangent at $(0; -1)$

Since $x = 0$ is not a solution, $x = \frac{26}{27}$ follows. The two summands are $\left(\frac{26}{27}, \frac{136}{27}\right)$. For the sought-after product, the following applies

$$\frac{3536}{729} = \left(\frac{17}{9}\right)^3 - \frac{17}{9}$$

Interpretation by I. Bashmakova

The mathematician considers the general case and interprets the equation as an elliptical curve, in which the coordinate axes are interchanged.

$$x(a - x) = y^3 - y \tag{21.2}$$

For $a = 6$, it is shown in Fig. 21.3. The following theorem holds for some elliptical curves: *If such a curve has two rational solutions, then a third can be obtained by intersecting the curve with the secant through the two given points. If the two rational points coincide, then the secant is replaced by the corresponding tangent.* Eq. (21.2) has a rational solution $(x|y) = (0| - 1)$. Through this point, a line can be drawn with the slope k

$$y = kx - 1$$

This explains the above linear approach. For this line to be tangent to curve (2), the intersection approach must yield a double solution

$$x(a - x) = (kx - 1)^3 - (kx - 1)$$

Calculation gives

$$ax - x^2 = k^3x^3 - 3k^2x^2 + 2kx$$

This equation certainly has a double solution (tangent case) if the linear terms are omitted. This condition shows

$$a = 2k \Rightarrow k = \frac{a}{2}$$

This is exactly what Diophantus does; for the given number 6, he sets $k = 3$. This gives the general solution

$$x = \frac{3k^2 - 1}{k^3}$$

The intersection of the tangent with the curve is $\left(\frac{26}{27} | \frac{17}{9}\right)$.

I. Bashmakova calls this approach the d*iophantine tangent method*. Even after fierce opposition from colleagues, Bashmakova sticks to her interpretation. A. Meskens rejects Bashmakova's suggestions in his book: *They are original interpretations, but have no connection to the historical development.* He writes:

> There is no doubt in our minds that each individual problem that we encounter in the *Arithmetica* was already solved before Diophantus. His Arithmetica is neither algebra, nor number theory, but an anthology of algorithmic problem solving.

Task IV, 27

To find two numbers, thus the product added to each results in a cubic number.

Solution: Diophantus sets the first number $8x$, the second $x^2 - 1$. The product of the two is $8x\left(x^2 - 1\right)$. Added to the first, $8x\left(x^2 - 1\right) + 8x = 8x^3$, results in a cubic number, as required. Added to the second number, it follows: $8x\left(x^2 - 1\right) + x^2 - 1 = 8x^3 + x^2 - 8x - 1$. This should be equal to a cubic number like $(2x - 1)^3$. Setting it equal delivers the approach

$$8x^3 + x^2 - 8x - 1 = (2x - 1)^3 \Rightarrow 13x^2 - 14x = 0; x > 0$$

Solution is therefore $x = \frac{14}{13}$; the two numbers are therefore $\left(\frac{112}{13}, \frac{27}{169}\right)$.

Task V, 9

13 is to be broken down into the sum of two squares, each of which is greater than 6.

Solution: Diophantus adds a small number to half in order to make a square : $\frac{13}{2} + \frac{1}{x^2}$. Since the quadruple is also a square, this delivers $4\left(\frac{13}{2} + \frac{1}{x^2}\right) = 26 + \frac{1}{y^2}$ with $y = \frac{x}{2}$. For the square sought after, Diophantus makes the approach

$$26 + \frac{1}{y^2} = \left(5 + \frac{1}{y}\right)^2$$

The solution of the quadratic equation provides $y = 10$ or $x = 20$. The term $\frac{13}{2} + \frac{1}{x^2}$ is therefore

$$\frac{13}{2} + \frac{1}{x^2} = \frac{13}{2} + \frac{1}{400} = \left(\frac{51}{20}\right)^2$$

Diophantus decomposes 13 into two squares and obtains $13 = 2^2 + 3^2$. He decomposes both roots into a sum or difference with $\frac{51}{20}$. This results in

$$3 = \frac{51}{20} + \frac{9}{20} \therefore 2 = \frac{51}{20} - \frac{11}{20} \tag{21.3}$$

After $\frac{51}{20}$ resolved, no equation for 13 results, but an approximation ((παρισοσοτες, almost equality)

$$\left(3 - \frac{9}{20}\right)^2 + \left(2 + \frac{11}{20}\right)^2 = 2\left(\frac{51}{20}\right)^2 = 13\frac{1}{200} \approx 13$$

With a new variable x, the approach for the sides $(3 - 9x), (2 - 11x)$ results from Eq. (21.3). The square decomposition is therefore

$$(3 - 9x)^2 + (2 + 11x)^2 = 13 \Rightarrow 202x^2 - 10x + 13 = 13$$

This provides $x = \frac{5}{101}$, the desired decomposition is finally

$$\left(\frac{257}{101}\right)^2 + \left(\frac{258}{101}\right)^2 = 13$$

21.3 From Diophantus' Book VI

Book VI of the *Arithmetica* contains some tasks in a geometric disguise. The numbering is given according to T. Heath; this edition differs in numbering from other editions.

Task VI, 2
To find a [right-angled] triangle so that the hypotenuse, added to the two catheti, results in a cubic number.

Solution: The triangle is set to $\left(4 - x^2; 4x; 4 + x^2\right)$ with $x < 2$. Addition of the hypotenuse and cathetus yields $(4 + x^2) + 4x = (x + 2)^2$. Since $x + 2 < 4$ must be and at the same time a cubic number, Diophantus chooses $x + 2 = \frac{27}{8}$ and thus $x = \frac{11}{8}$. The sought triangle therefore has the sides $\left(\frac{135}{64}; \frac{11}{2}; \frac{377}{64}\right)$ or integer (135; 352; 377). It is

$$377 + 135 = 8^3 \therefore 377 + 352 = 9^3$$

Task VI, 6
To find for a [right-angled] triangle so that the area, added to one of the catheti, results in a given number (7).

Solution: The triangle is first set similar to the triangle (3x; 4x; 5x), the area is thus $6x^2$. Added to the smaller cathetus, it follows $6x^2 + 3x = 7$. This has no

rational solution. Therefore, the similarity to the triangle $(24x;\ 7x;\ 25x)$ is chosen with the area $84x^2$. It results in the condition $84x^2 + 7x = 7$ with the solution $x = \frac{1}{4}$. The sought triangle is thus $\left(6;\ \frac{7}{4};\ \frac{25}{4}\right)$.

Task VI, 7
To find a

[right-angled] triangle such that the difference of the area and one of the catheti results in a given number (7).

Solution: The triangle is set up as $(24x;\ 7x;\ 25x)$ as in the previous exercise. This results in the condition $84x^2 - 7x = 7$ with the solution $x = \frac{1}{3}$. The sought triangle is therefore $\left(8;\ \frac{7}{3};\ \frac{25}{3}\right)$.

Task VI, 10
To find a [right-angled] triangle such that the sum of the area, hypotenuse and one of the cathetus results in a given number (4).

Solution: The approach $(28x;\ 45x;\ 53x)$ provides the condition $630x^2 + (53 + 28)x = 4$. Solution is $x = \frac{4}{105}$. The sought triangle is therefore $\left(\frac{16}{15};\ \frac{12}{7};\ \frac{212}{105}\right)$. The analogous approach for the second cathetus does not yield a rational solution.

Task VI, 16
To find a [right-angled] triangle in which the angle bisector (of an acute angle) has a rational length.

Solution: Diophantus sets the triangle CDB as similar to the triangle (3; 4; 5) without further comment (Fig. 21.4). CD is the angle bisector of the triangle ABC. The side $|AB|$ is considered to be a multiple of the length 3; thus follows

$$|AB| = 3y \Rightarrow |AD| = |AB| - |BD| = 3y - 3x = 3(y - x).$$

According to Euclid [VI, 3], each angle bisector divides the opposite side in the ratio of the adjacent sides. Thus,

$$\frac{|AC|}{|BC|} = \frac{|AD|}{|BD|} \Rightarrow |AC| = |BC| \cdot \frac{|AD|}{|BD|} = 4x \cdot \frac{3(y - x)}{3x} = 4(y - x)$$

Application of the Pythagoras to the triangle $\triangle ABC$ yields

$$|AC|^2 + |BC|^2 = |AB|^2 \Rightarrow 16(y - x)^2 = 9y^2 + 16x^2$$

Simplifying yields for $y > 0$

$$16y^2 - 32xy + 16x^2 = 9y^2 + 16x^2 \Rightarrow y(7y - 32x) = 0 \Rightarrow x = \frac{7}{32}y$$

In order for the multiple x to become an integer, one sets $y = 32$ and thus obtains $x = 7$. The sought triangle therefore has the sides $|AB| = 3y = 96, |BC| = 4x = 28$

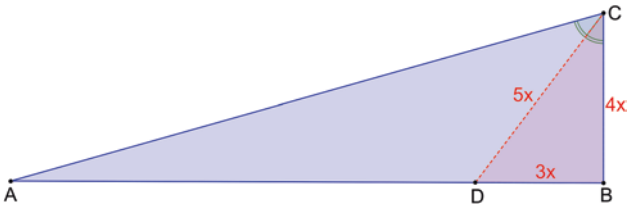

Fig. 21.4 To Diophantus VI, 16

and $|AC| = 4(y - x) = 4(32 - 7) = 100$. The angle bisector is $|CD| = 5x = 35$. The sought triangle ABC is (96; 28; 100); it is similar to (24; 7; 25).

21.4 From Diophantus' Books in Arabic

The four books of Diophantus' *Arithmetica,* which have been preserved only in Arabic translation[12] , are numbered as IV to VII. Since this numbering overlaps with the Greek one, the four Arabic books are referred to here as A to D. If you sort the tasks by difficulty, the order I, II, III, IV, A, B, C, D, V, VI results. J. Sesiano thinks that Arabic Book IV continues Greek Book III; he puts Greek Books IV-VI in quotation marks. This results in the unusual numbering I, II, III, IV, V, VI, "IV", "V", "VI".

Task A1
Two find two numbers such that the sum of their cubes is a square.

Solution: The equation is considered

$$x^3 + y^3 = z^2$$

Diophantus makes the linear ansatz again $y = 2x$ and $z = 6x$. Substituting yields the equation

$$x^3 + 8x^3 = 36x^2 \Rightarrow x = 4$$

This gives $y = 8$ and $z = 24$. The numbers we are looking for are 4, 8, and $4^3 + 8^3 = 24^2$.

A general approach is: $y = mx, z = nx$. This leads to $x^3\left(1 + m^3\right) = n^2 \Rightarrow x = \frac{n^2}{1+m^3}$. There are numerous other integer solutions for (n, m):

n	m	x	y	z
2	1	2	2	4

[12] Sesiano J. (ed.): Books IV to VII of Diophantus' *Arithmetica* in the Arabic Translation Attributed to Qusṭā ibn Lūqā, Springer 1982.

n	m	x	y	z
4	1	8	8	16
6	1	18	18	108
6	2	4	8	24
8	1	32	32	256
9	2	9	18	81

Task A25
To find two numbers , one a square, the other a cubic number, so that the sum of their squares is a square again.

Solution: If you set the sum of squares equal to z^2, you have to solve

$$\left(x^2\right)^2 + \left(y^3\right)^2 = z^2$$

Diophantus now sets $x = 2y$ and $z = ky^2$. With this you get

$$16y^4 + y^6 = k^2y^4 \Rightarrow y^6 = \left(k^2 - 16\right)y^4 \Rightarrow y^2 = k^2 - 16$$

A square that, reduced by 16, is a square again, is 25. From this it follows that $y = 3$. The sought cubic number is therefore $y^3 = 27$, the sought square $(2y)^2 = 36$. In fact,

$$\left(6^2\right)^2 + \left(3^3\right)^2 = 45^2$$

Task B7
To find two numbers so that their sum and the sum of their cubes are equal to two given numbers. Diophantus chooses the numbers $a = 20, b = 2240$.

Solution: The following system is to be solved

$$x + y = 20 \therefore x^3 + y^3 = 2240$$

As usual, Diophantus sets the summands $(10 \pm z)$. Substitution gives

$$(10 + z)^3 + (10 - z)^3 = 2240$$

Diophantus explains in detail in words the formula $(a + b)^3$ and simplifies the formula to

$$2000 + 60z^2 = 2240 \Rightarrow z = 2$$

The numbers you are looking for are therefore $x = 12; y = 8$. There are no other integer solutions.

Task B8
To find two numbers so that their difference and the difference of their cubes are equal to two given numbers. Diophantus chooses the numbers $a = 10, b = 2170$.

Solution: So the following system is to be solved

$$x - y = 10 \therefore x^3 - y^3 = 2170$$

Substitution of $y = x - 10$ in the second equation gives simplified

$$x^3 - (x - 10)^3 = 2170 \Rightarrow x^2 - 10x = 39$$

The solution is therefore $x = 13; y = 3$. There are no other integer solutions.

Task B9
A given number (a) is to be divided into two summands in such a way that the sum of their cubes is a given multiple of the square of their difference. Diophantus chooses the numbers $a = 20, b = 140$.

Solution: The following system is to be solved

$$x + y = 20 \therefore x^3 + y^3 = 140(x - y)^2$$

Inserting $y = 20 - x$ into the second equation gives simplified

$$x^3 + (20 - x)^3 = 140(2x - 20)^2 \Rightarrow x^2 - 20x + 96 = 0$$

The solution is therefore $x = 12, y = 8$.

Task C1
Two numbers are in the ratio 2:1. To find the numbers so such that one is a square, the other a cubic number and the sum of the squares of both is again a square.

Solution: It should hold

$$\left(x^2\right)^2 + \left(y^3\right)^2 = z^2 \therefore \frac{y}{x} = \frac{2}{1}$$

To compensate for the sixth power, it is set $z = 10x^3$. Inserting both conditions gives

$$x^4 + 64x^6 = 100x^6 \Rightarrow x = \frac{1}{6}$$

This gives $y = \frac{2}{6}$ and $z = \frac{10}{216}$. In fact, it holds

$$\left(\frac{1}{6}\right)^4 + \left(\frac{2}{6}\right)^6 = \frac{100}{46656} = \left(\frac{10}{216}\right)^2$$

Task D17
The sum of three squares should be a square again, it should apply: The square of the first number is equal to the second, the square of the second number is equal to the third.

$$x^2 + y^2 + z^2 = w^2 \therefore x^2 = y \therefore y^2 = z$$

Inserting gives $x^2 + x^4 + x^8 = w^2$. With the approach $w = \left(x^4 + \frac{1}{2}\right)^2$ the eighth power is eliminated; it follows $x^2 = \frac{1}{4}$. The sum sought is thus

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{16}\right)^2 = \left(\frac{9}{16}\right)^2$$

21.5 To the Mathematics of Diophantus'

In his introduction, he gives a precise description of how to solve a (linear) equation (Heath, p.131):

> If, in solving a problem, one comes to an equation which consists of the same general expressions, but so that the coefficients on both sides are unequal, one must subtract like from like until one term is equal to a term. But if on one or both sides there are terms which are to be subtracted, one must add these terms to both sides until only terms which are to be added remain on both sides. Then one must again subtract like from like until only one term remains on each side.

Diophantus is characterized by the use of special expressions for the unknown and its powers up to the sixth: *arithmos, dynamis* (Δ), *kybos* (K), *dynamodynamis* (ΔΔ), *dynamokybos* (ΔK), *kybokybos* (KK). He writes the unknown with a sign similar to "ς". The form of the natural numbers follows the alphabet, each with an accent: $\grave{\alpha} = 1$, etc.

Accordingly, he chooses names for the reciprocal values $\frac{1}{x}$ to $\frac{1}{x^6}$ that are similar to the names of fractions: *arithmoston, dynamoston, kyboston* etc. For square numbers he distinguishes between *tetragonos* and *dynamis:* The former is any square number, *dynamis* is the unknown in the second power; on the other hand, *kybos* means both x^3 and cubic number in general. For fractions he writes either the denominator over the numerator or next to each other with the reciprocal of the denominator; an example is

$$\frac{3}{2} = \frac{\beta}{\gamma} \text{ or } \beta^{\times}\gamma$$

His signs are simply abbreviations of the names; he calls subtraction *leipsis* (λεῖψις), his minus sign is to be read in the manuscripts like "ʌ". Addition is called *hyparxis* (ὕπαρξις); he has no sign for "plus", he places the summands next to each other, as is the case with Egyptian fractions. The unit is called *mona* and is written as M and the constant 5 as Mε. The equality sign is written as "ι^{σ}," abbreviated for "ἴσοε". All terms are arranged so that the numbers to be added are in front, followed by the numbers to be subtracted. A term like $(x^3 + 3x - 2x^2 - 4)$ he writes as:

$K^{\gamma}\alpha\varsigma\gamma \wedge \Delta^{\gamma}\beta\mathrm{M}\delta$

The following **algebraic knowledge** can be found in Diophantus, here written in modern form:

1. $x^2 + y^2 \pm 2xy$ is a square (II, 31).

2. $\left(\frac{x-y}{2}\right)^2 + xy$ is a square (II, 35).
3. $x^3 + y^3 + 3x^2y + 3xy^2$ is a perfect cube.
4. Any number that is the sum of two squares can be expressed as the sum of two (rational) squares in an infinite number of ways (II, 8) (II, 9).
5. A number of the form $4n + 3$ can never be the sum of two (integer) squares (V,12).
6. Every prime number of the form $4n + 1$ can be expressed as the sum of two (integer) squares (V,12).
7. A number of the form $8n + 7$ can never be the sum of three (integer) squares (V,12); the precise form is given by Fermat as $4^a(8n + 7)$ with $a \in \mathbb{N}_0$.
8. Any number is the sum of at most four (rational) squares. Examples are

$$13 = \left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2 + \left(\frac{9}{5}\right)^2 + \left(\frac{12}{5}\right)^2$$

$$30 = 1^2 + 2^2 + 3^2 + 4^2 = 2^2 + 3^2 + \left(\frac{1016}{349}\right)^2 + \left(\frac{1019}{349}\right)^2.$$

9. $\frac{a+b}{2} + \frac{a-b}{2} = a \therefore \frac{a+b}{2} - \frac{a-b}{2} = b$ (I, 18) (I, 19)

   The following three sentences are probably examples of lost Porismen.
10. If three numbers are different, then the sum of their differences is twice the difference of the largest and smallest number (IV, 14) (IV, 25):

$$a > b > c \Rightarrow (a - b) + (b - c) = 2(a - c)$$

11. If three numbers have the same difference, then the sum of the largest and smallest is twice the middle number (I, 39).

$$a - b = b - c \Rightarrow a + c = 2b$$

12. If three numbers have the same difference, this also applies to the sums of two of these numbers.

$$a - b = b - c \Rightarrow (a + c) - (b + c) = (a + b) - (a + c)$$
$$\Rightarrow (a + b) - (a + c) = (a + c) - (b + c)$$

13. The sum of two squares, increased or decreased by twice the product, results in a square number (II, 39) (Euclid II, 4).

$$\left(a^2 + b^2\right) \pm 2ab = (a \pm b)^2$$

14. The double sum of two squares exceeds the square of the sum by a square number (I, 28) (Euclid II, 10).

$$2\left(a^2 + b^2\right) = (a + b)^2 + (a - b)^2$$

15. The fourfold product of two numbers increased by the square of the difference yields a square number (I, 30) (Euclid II, 8).

$$4ab + (a-b)^2 = (a+b)^2$$

16. The product of two consecutive square numbers increased by their sum yields a square number (III, 15).

$$n^2(n+1)^2 + n^2 + (n+1)^2 = \left[n^2 + (n+1)\right]^2$$

17. The difference of two squares can always be decomposed into a product (II, 11).

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

In (II, 34) Diophantus uses this sentence in the following form: If you add the square of the mean of both factors to the product of two numbers, you get a square number again.

18. The difference of two cubic numbers is also the sum of two (rational) cubic numbers (V, 16). Diophantus does not provide an example here.

A solution is found by Vieta, actually François Viète, in his work *Zetetica* IV, 18–20:

$$a^3 - b^3 = x^3 + y^3 \Rightarrow x = \frac{a\left(a^3 - 2b^3\right)}{a^3 + b^3}; y = \frac{b\left(2a^3 - b^3\right)}{a^3 + b^3}$$

Examples are:

$$6^3 - 3^3 = 4^3 + 5^3 \therefore 8^3 - 4^3 = \left(\frac{16}{3}\right)^3 + \left(\frac{20}{3}\right)^3$$

The transformations of formal fractions in Book IV are also remarkable. In Problem (IV, 39), the following equation is solved:

$$3x^2 + 12x + 9 = (3 - nx)^2 \Rightarrow x = \frac{12 + 6n}{n^2 - 3}$$

The verbal description is: "x is six times a number, increased by 12, which is divided by the difference between the square of the number and 3."

Some deformations, here in modern form:

19. $\frac{3x}{x-3} \cdot \frac{4x}{x-4} = \frac{12x^2}{x^2+12-7x}$ (IV, 36)
20. $\frac{96}{x^4+36-12x^2} - \frac{12}{6-x^2} = \frac{2x^2+24}{x^4+36-12x^2}$ (IV, 13)
21. $\frac{8}{x^2+x} + 1 = \frac{x^2+x+8}{x^2+x}$ (IV, 25)
22. $\left(4x^2 + 6x + 2\right) = (x+1)(4x+2)$ (IV, 19)

The last formula shows that Diophantus could decompose terms (like polynomials). This could explain why he simply writes in task (VI, 19) that it applies: $x^3 + 3x - 3x^2 - 1 = x^2 + 2x + 3$, it follows $x = 4$. Maybe he could transform:

$$x^3 + x = 4x^2 + 4 \Rightarrow x\left(x^2 + 1\right) = 4\left(x^2 + 1\right) \Rightarrow x = 4$$

23. The following identity of square sums can already be found in Diophantus (III, 19), but was only proven by Leonardo of Pisa.

$$\left(a^2 + b^2\right)\left(c^2 + d^2\right) = (ac \pm bd)^2 + (ad \mp bc)^2$$

The identity is also named after Fibonacci or Brahmagupta (598–668). This yields the following statement: *The product of two numbers, both of which are the sum of two squares, can always be decomposed into the sum of two squares* (III, 19). A. Weil[13] suspects that this statement is the second mentioned Porism. Diophantus uses the statement in connection with two right-angled triangles, as explained by L.E. Dickson[14]If (a, b) or (c, d) are the pairs of catheti, then four more pairs of catheti can be obtained from $(ac \pm bd, ad \mp bc)$ if

$$\frac{c}{d} \notin \left\{\frac{a}{b}; \frac{b}{a}; \frac{a \pm b}{a \mp b}\right\}$$

We consider the triangles (3; 4; 5) and (5; 12; 13).. If we multiply the catheti of the first triangle by the hypotenuse of the second and vice versa, we obtain the number pairs {(39; 52); (25; 60)}. This way we have found two triangles with hypotenuse 65: (39; 52; 65), (25; 60; 65). This hypotenuse can be decomposed according to the construction:

$$65 = 5 \cdot 13 = \left(2^2 + 1^2\right)\left(3^2 + 2^2\right)$$

From this product, two further decompositions can be derived from the above-mentioned identity of Fibonacci:

$$\left(2^2 + 1^2\right)\left(3^2 + 2^2\right) = (6 + 2)^2 + (4 + 3)^2 = 8^2 + 7^2$$

If you consider the right side as the hypotenuse z, then the catheters result from the Pythagoras triple:

$$x = 2ab \therefore y = a^2 - b^2 \therefore z = a^2 + b^2$$

You get:

$$(a; b) = (8; 1) \Rightarrow x = 16; y = 63$$
$$(a; b) = (7; 4) \Rightarrow x = 56; y = 33$$

From this you gain two more triangles (16; 63; 65), (33; 56; 65). In total, four right-angled triangles with hypotenuse 65 have been found:

$$(39; 52; 65), (25; 60; 65), (16; 63; 65)(33; 56; 65)$$

[13] Weil A.: Zahlentheorie: Ein Gang durch die Geschichte von Hammurapi bis Legendre, Birkhäuser 1992, S. 11.

[14] Dickson L.E.: History of the Theory of Numbers, Volume II, Diophantine Analysis, Dover 2005, S. 165, 225.

21.6 Linear Diophantine Equation

Under a linear diophantine equation one understands *today* an equation linear in several unknowns, for which only integer solutions are sought, in contrast to Diophantus, who always allowed rational solutions. In Diophantus, indeterminate linear equations occur only occasionally, since usually a number is given. The first solutions of linear equations by means of elimination were due to Brahmagupta and later Bhaskara II.

A linear diophantine equation with two unknowns x, y has the form

$$ax + by = c;\ a, b, c \in \mathbb{Z}$$

According to the Theorem by Bachet the equation $ax + by = c$ always has a solution if the following is true: $gcd(a, b)$ is a divisor of c. First, one looks for a specific solution of $ax + by = gcd(a, b)$. This implies that the $gcd(a, b)$ can always be represented as a linear combination of the two numbers a, b. As an example, the following is chosen:

$$4x + 7y = 5\ (1)$$

The equation is solvable because of $gcd(4, 7) = 1|5$. A solution of $4x + 7y = 1\ (2)$ results from the extended Euclidean algorithm

$$\begin{aligned} 7 &= 1 \cdot 4 + 3 \Rightarrow 3 = 7 - 1 \cdot 4 \\ 4 &= 1 \cdot 3 + 1 \Rightarrow 1 = 4 - 1 \cdot 3 \\ 3 &= 1 \cdot 3 + \boxed{0} \end{aligned}$$

Back calculation gives

$$1 = 4 - 1 \cdot 3 = 4 - 1(7 - 1 \cdot 4) = 2 \cdot 4 + (-1) \cdot 7$$

This means that $(x = 2; y = -1)$ is a possible solution of (2). Multiplication by 5 gives $4 \cdot 10 + 7 \cdot (-5) = 5$. A specific solution of (1) is therefore $x_0 = 10, y_0 = -5$. The general solution must contain a parameter t; therefore, one sets $x = x_0 + t \cdot b$ or $y = y_0 - t \cdot a$. Re-calculation confirms this choice

$$ax + by = a(x_0 + t \cdot b) + b(y_0 - t \cdot a) = ax_0 + tab + by_0 - tab = ax_0 + by_0 = c$$

The general integer parameter solution of our example is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} t + \begin{pmatrix} 10 \\ -5 \end{pmatrix}; t \in \mathbb{Z}$$

However, the representation is not unique; it can also be represented as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} t + \begin{pmatrix} -4 \\ 3 \end{pmatrix}; t \in \mathbb{Z}$$

An example of a linear diophantine equation with three variables is:

$$3x + 4y - 5z = 6$$

Here two parameters are required. In order for the solution to be an integer, one sets $y = 3s$; $z = 3t$. So it follows that

$$3x = 6 - 4y + 5z = 6 - 12s + 15t \Rightarrow x = 2 - 4s + 5t$$

The general 2-parameter solution is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} s + \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} t + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}; s, t \in \mathbb{Z}$$

Linear diophantine equations have been popular as problems since ancient times and the Middle Ages. Probably the best known is the Chinese 100-bird problem by Chang *Ch'iu-Chien*:

> A rooster costs 5 *sapek*, a hen 3 *sapek*, 3 chicks cost 1 *sapek*. How many birds of each kind can be purchased for the sum of 100 *sapek*, if a total of 100 animals are to be bought?

The diophantine system results:

$$x + y + z = 100 \therefore 5x + 3y + \frac{1}{3}z = 100$$

Elimination of z gives: $7x + 4y = 100$. The system is underdetermined, we set the parameter $t = x/4$. This gives $y = 25 - 7t$ and after insertion $z = 75 + 3t$. The general (integer) solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 25 \\ 75 \end{pmatrix} + \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} t; t \in \mathbb{Z}$$

Positive solutions ($y > 0$) exist for $t \in \{1; 2; 3\}$; these are $\{4; 18; 78\}$, $\{8; 11; 81\}$ and $\{12; 4; 84\}$, as indicated by Ch'iu-Chien.

21.7 Outlook

Numerous number-theoretical questions that can be found in Diophantus have inspired subsequent generations to undertake a variety of research. The famous 10th Problem of Hilbert was not solved until 1970 by J. Matijassewitch (proof in 288 pages); he was able to prove that the set of solution sets of Diophantine equations is equipotent with the set of recursively enumerable sets and therefore undecidable. The aforementioned "Great Theorem of Fermat" played a significant role in the development of number theory. That the Fermat problem has at most finitely many solutions could not be proven until 1983 by GA. Wiles (proof in 90 pages).

The history of the reception of Diophantus's work is extensive. Many famous mathematicians were involved in the translation and editing, such as M. Planudes, Fibonacci, Regiomontanus, R. Bombelli, Xylander, Vieta and finally C.-G. Bachet (de Meziriac). The latter produced a exemplary edition in 1621, from which P. Fermat[15] acquired a copy and noted important findings in the margins; this was the birth of (modern) number theory. The whole story can be read in A. Meskens[16] .

Some further questions are addressed here. G.J. Jacobi judged very accurately about Diophantus:

> But Diophantus will always be remembered for giving the first impetus to the deeper properties and relationships of numbers, which were opened up by the beautiful research of newer mathematics.

21.7.1 The Problem of Congruent Numbers

The starting point of the problem was task (VI, 20), which, according to T. Heath, is a later insertion by C.-G. Bachet (de Meziriac).

Task VI, 20: To find a (right-angled) triangle so that the sum of the area and one of the catheti is a cubic number, the circumference is a square number. [Solution: $\left(\frac{16}{9}; \frac{63}{9}; \frac{65}{9}\right)$].

P. Fermat wrote on the margin of his Diophantus edition for this task: *The area of a right-angled triangle with rational sides can never be a square!*

This led to the question of which numbers the area of such a triangle can be. We define: A number is called *congruent* if it is the integer area of a right-angled triangle with rational sides. The problem was already mentioned in an Arabic manuscript by Mohammed Ben al-Ḥokain (tenth century) dealing with Pythagorean triples. The problem can be formulated as

$$a^2 + b^2 = c^2; a, b, c \in \mathbb{Q} \therefore \frac{1}{2}ab = n \in \mathbb{N}$$

By means of the substitution

$$a = \frac{1}{y}\left(x^2 - n^2\right); b = \frac{1}{y}2nx; c = \frac{1}{y}\left(x^2 + n^2\right)$$

the task can be reduced to solving the Diophantine equation $y^2 = x^3 - n^2x$, which belongs to the class of elliptic curves (according to Ash/Gross[17]). It is conjectured that a natural number $n \geq 5$ is congruent if it is square-free and $\{5, 6, 7 \bmod 8\}$.

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[15] Müller M.(ed.): Bemerkungen zu Diophant von Pierre Fermat, Akademische Verlagsgesellschaft Leipzig 1932.

[16] Meskens A.: Travelling Mathematics – The Fate von Diophantos' Arithmetic, Birkhäuser 2010.

[17] Ash A., Gross R.: Elliptic Tales, Curves, Counting and Number Theory, Princeton University 2012, p. 241.

All congruent numbers up to 20 can be found in the following table:

Area A	a	b	c
5	$\frac{3}{2}$	$\frac{20}{3}$	$\frac{41}{6}$
6	3	4	5
7	$\frac{35}{12}$	$\frac{24}{5}$	$\frac{337}{60}$
13	$\frac{780}{323}$	$\frac{323}{30}$	$\frac{106921}{9690}$
14	$\frac{8}{3}$	$\frac{63}{6}$	$\frac{65}{6}$
15	4	$\frac{15}{2}$	$\frac{17}{2}$
20	3	$\frac{40}{3}$	$\frac{41}{3}$

21.7.2 Representability by the Sum of Two Cubes

A question raised by J.J. Sylvester is which (natural) numbers are representable as the sum of two cubes:

$$x^3 + y^3 = n$$

He conjectured that n is always representable if n is prime and $n = 4 \bmod 9$. This has not yet been proven. However, P. Satgo was able to prove a very similar theorem in 1986:

If $n/2$ is prime and $n = 4 \bmod 9$, then n is representable as $x^3 + y^3$. An example is:

$$22 = \left(\frac{17299}{9954}\right)^3 + \left(\frac{25469}{9954}\right)^3$$

21.7.3 Representability by the Sum of Three Cubes

Inspired by the task, which sum of three cubes is a square, the question arose: Which natural number under 100 can be represented as the sum of three cubes?

$$x^3 + y^3 + z^3 = n$$

For each cube of an integer n it holds:

$$n^3 = \begin{cases} -1 \\ 0 \\ 1 \end{cases} \bmod 9$$

Therefore, the sum of three cubic numbers cannot take on the values $\{4; 5 \bmod 9\}$. This leaves 22 possibilities up to 100. Examples of decompositions of the remaining 78 numbers with two or one solution are:

$$x^3 + y^3 + z^3 = 18 \Rightarrow (x, y, z) = (3; -2; -1)$$

$$x^3 + y^3 + z^3 = 29 \Rightarrow (x, y, z) = (3; 1; 1) \vee (x, y, z) = (4; -3; -2)$$

$$x^3 + y^3 + z^3 = 30 \Rightarrow (x, y, z) = (2\,220\,422\,932; -2\,218\,88\,517; -283\,059\,965)$$

No solution had been found by summer 2019 for the numbers $\{33; 42\}$. The next step was achieved by A. Booker from the University of Bristol; he found the following decomposition into three cubes for the 33 numbers:

$$x^3 + y^3 + z^3 = 33 \Rightarrow (x, y, z)$$
$$= (8866128975287528; -8778405442862239; -2736111468807040)$$

The problem for the number 42 seemed to be open until the press release of the DMV (German Mathematical Association) on September 9, 2019. In cooperation with Booker, A. Sutherland from MIT (Massachusetts Institute of Technology) was able to find a solution by using a network of half a million home PCs:

$$x^3 + y^3 + z^3 = 42 \Rightarrow (x, y, z)$$
$$= (-80538738812075974; 80435758145817515; 12602123297335631)$$

21.7.4 Representability by the Sum of Two Squares

Diophantus's problem (II, 8) led to the question: Which numbers can be represented as the sum of two squares? In problem (VI, 15) Diophantus states that the number 15 cannot be decomposed into the sum of two squares.

Here the theorem of P. Fermat applies: Every natural number is exactly the sum of two (integer) squares if the prime numbers occurring in the factorization are of the form (3 mod 4) and occur an even number of times.

Examples The number $125 = 5^3$ has no prime factor of the form (3 mod 4), so it is decomposable into square sums

$$125 = 5^2 + 10^2 = 2^2 + 11^2$$

The number $306 = 2 \cdot 3^2 \cdot 17$ has the prime factor 3 twice, so it is decomposable

$$306 = 9^2 + 15^2$$

The theorem of G.J. *Jacobi* (1828) allows to determine the number of decompositions into squares of whole numbers, including the permutations and sign changes. There is the number:

$$4\left[d_{1,4}(n) - d_{3,4}(n)\right]$$

It is $d_{r,4}(n)$ the number of all divisors d of the number n ($1 \leq d \leq n$), for which is valid: $d \bmod 4 = r$. If one takes into account that there are always four combinations of signs

$$a^2 + b^2 = (-a)^2 + b^2 = a^2 + (-b)^2 = (-a)^2 + (-b)^2$$

and two combinations ($a \leftrightarrow b$) of permutations, the number of decompositions remains

$$\frac{1}{2}\left[d_{1,4}(n) - d_{3,4}(n)\right]$$

Examples The number 325 has the divisor set $\{1, 5, 13, 25, 65, 325\}$, all divisors are 1 mod 4. Thus: $d_{1,4}(325) = 6$; $d_{3,4}(325) = 0$. Thus, only three decompositions remain . These are

$$325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2$$

For the number 306 considered above, it holds: Of the twelve divisors, four of them are equal (1 mod 4) and two are equal (3 mod 4). There are only $\frac{1}{2}\left[d_{1,4}(306) - d_{3,4}(306)\right] = \frac{1}{2}(4 - 2) = 1$ decompositions in sums of squares of natural numbers (as above)

$$306 = 9^2 + 15^2$$

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Pappus of Alexandria

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Pappos (Παππος ο Αλεξανδρευς) lived approximately 300 BC and probably worked in Alexandria. The determination of his life data is difficult, as there are hardly any comments from later generations. He himself states that he observed the eclipse of October 18, 320 BC. The attribution of the eclipse is carried out by A. Rome[1]; the almost total eclipse of 346 would also have been possible. Pappos would certainly have reported this experience.

The Suda writes that Pappos was a contemporary of Theon of Alexandria; this is just barely possible if the life of the older Theon is set at 335–405 BC. In a scholion to Theon's chronological table, under the keyword Diocletian (reign 284–305), it is noted *In this time lived Pappos,* which is still compatible with the above eclipse date.

Pappos lived as a follower of the old Platonic mathematics in a turbulent century. The persecutions of Christians had begun in 302/3 AD; by 395 Christianity was already the state religion. Mathematics was pushed back as a pagan science and had no future. Pappos could only look back and try to preserve as much of the mathematical tradition in his books. S. Cuomo[2] writes:

> Pappos' *Collectio* is considered as the *requiem of Greek mathematics*.

Pappos' main work *Collectio* (συναγωγαι μαθεματικε) is a *unique* compilation of Greek geometry in eight books, which was probably created over a longer period of time. Preceding it in time is his *commentary on the Almagest;* in book VIII, 46 he mentions a theorem by Archimedes, which he had proven independently of this:

[1] A. Rome: Commentaries by Pappus and Theon of Alexandria on the Almagest I, Rome 1931.

[2] Cuomo S.: Pappus of Alexandria and the Mathematics of Late Antiquity, Cambridge 2007, p. 249.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_22

> The rectangle of the radius and circumference U of a circle is equal to [the area of] the double circle: $rU = 2\pi r^2 = 2A$.

The *Collectio*, probably intended for 12 books, contains a wealth of theorems, problems and commentary on Greek mathematics that would otherwise have been lost. Pappus, at the end of a long era of Greek-Hellenistic mathematics, provides here a survey whose value can hardly be overestimated.

In many books Pappus is only classified as a second-rate mathematician because he does not arrange the theorems in his books with the same purposefulness as Euclid's *Elements*. But in addition to numerous lemmas on Apollonius, Pappus has made original contributions to geometry, which 1200 years later is called projective. These are the theorems on the complete quadrilateral and the figure named after him, the Pappus figure. Therefore, he deserves a rightful place in the history of mathematics. The seventh and eighth book of his *Collectio* Pappus dedicated to his son Hermodoros; this is probably identical to the Hermodoros who, according to Proklos's statement, ran a school of mathematics in Alexandria. Of his numerous works, only parts of his commentary on Ptolemy and two books of his remarks on Euclid have survived, in addition to the *Collectio*.

The work *Collectio* is preserved as *Codex Vaticanus* gr. 218. The standard edition comes from by Friedrich Hultsch[3]. A German edition was published by Carl Immanuel Gerhardt[4]. Newer editions in English are by Heike Sefrin-Weis[5] and Alexander Jones[6].

Of the eight books, only five are completely preserved. Book I is completely missing; it probably only contained calculation rules. Book II is only preserved from Theorem 14; it contains a number system of Apollonius for values greater than 10,000. Book III contains the theorems on means, the paradoxes of an otherwise unknown Erykinos and theorems on the inscribing of the Platonic solids in a sphere, which deviates from the Euclidean representation.

Book IV contains a wealth of interesting individual topics, a generalization of Pythagoras' theorem, and important supplements to Archimedes: theorems on the Arbelos figure, angle trisection using a spiral, and some special cases from the Book of Lemmas. Furthermore, he gives a definition of a parabola using a focal line and focus, which is not found in Apollonius.

Book V is concerned in the first part with isoperimetric curves (i.e. curves of equal circumference); here it is shown that the circle has the largest area of all closed curves of equal circumference. An analogous theorem states that the sphere has the largest volume of all bodies of given surface area. Furthermore, it contains an almost literary contribution on the *cleverness of bees.* Furthermore, he

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[3] Hultsch F. (ed.): Pappi Alexandrini Collectionis, 3 volumes, Berlin 1876–1878.

[4] Gerhardt C.I. (ed.): Die Sammlung des Pappus von Alexandrien, 2 volumes, Halle 1871, Eisleben 1875.

[5] Sefrin-Weis H.(ed.): Pappus of Alexandria: Book IIII of the Collection, Springer 2010.

[6] Jones A. (ed.): Mathematical Collections, Book VII, Springer, 1986.

discusses 13 semi-regular bodies, which he attributes to Archimedes and which are therefore later called Archimedean bodies; Kepler discusses them in the second book of his *Harmonices Mundi.* Book VI contains astronomical contributions to various authors; these are not our topic.

Book VII is unique; it contains a wealth of valuable information about works that have not been transmitted. These are the *Data* and *Porismen* of Euclid, the *Book of Surface Decompositions,* the B*ook of Inclinations,* the *Book of Contacts* and the *Book of (geometric) Loci.* It also provides numerous lemmas to the *Conica* of Apollonius, in particular also to the lost Book VIII. Sentence (VII, 238) defines a parabola using the focal point and the directrix, lemma (VII, 129) confirms the invariance of the cross-ration of four points under projective transfomations.

The Book of Loci contains a criterion for testing whether four points A, B, C, D are collinear.

$$|AD|^2|BC| + |BD|^2|CA| + |CD|^2|AB| + |BC||CA||AB| = 0$$

The theorem was later proved by R. Simson, but named after M. Stewart. The preface of this book contains the famous rule later named after P. Guldin (1557–1643) .

The *Book of Porismen* contained in the second part contains in Theorem 129 the preservation of the double proportion in projective mappings, in Theorem 130 the complete quadrilateral with the invariance of the harmonic division. Theorem 139 provides the famous theorem about the hexagon figure of Pappos.

Book VIII was only partially preserved and includes mechanical problems with wheel and axle, lever, pulley, roller and screw. It also contains the construction of conic sections given five points and the inscribing of seven regular hexagons in a circle.

22.1 From Book VII of the Collectio

Pappus VII, 91

A small exercise is the following (Fig. 22.1):

Given are two semicircles, whose diameters lie on the line AF. The first semicircle over AC has the center D, the second the diameter DF. An arbitrary line through F intersects the first circle in the points B resp. G, the second in the point E. It is to be shown that $|BE| = |EG|$.

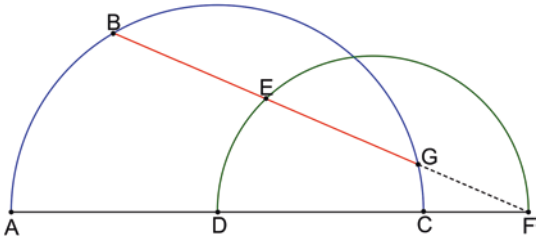

Fig. 22.1 Figure for Pappos VII, 91

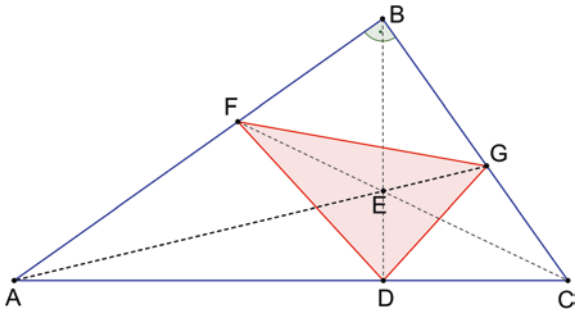

Fig. 22.2 Figure for the lemma Analoumes

Proof: $\measuredangle DEG$ is a right angle, since E lies on the Thales circle upon DF. $\triangle BDG$ is isosceles, since $|BD| = |DG|$. The line DE is thus the altitude in $\triangle BDG$ and at the same time the perpendicular bisector of side BG. Thus, the claim $|BE| = |EG|$.

Pappos VII, Conclusion

The conclusion of Book VII is in all manuscripts a lemma, which *Analuomes* is called and is obviously highlighted by Pappus. It states:

If in the right-angled triangle $\triangle ABC$ the two catheti AB and BC are divided by the points F and G in the same ratio as the ratio of the catheti, then the cevians AG and FC intersect in the point E so that BE is perpendicular to AC (Fig. 22.2).

By assumption, in the $\triangle ABC$:

$$\frac{|AF|}{|FB|} = \frac{|BG|}{|GC|} = \frac{|AB|}{|BC|}$$

Pappos concludes from the assumption

$$\frac{|AF|}{|FB|} = \frac{|BG|}{|GC|} \therefore \frac{|AB|}{|FB|} = \frac{|BC|}{|GC|} \underset{Div.}{\Rightarrow} \frac{|AB|}{|BC|} = \frac{|FB|}{|GC|}$$

Together with the assumption $\frac{|AB|}{|BC|} = \frac{|BG|}{|GC|}$ he concludes $|FB| = |BG|$.

This shows that $\triangle BFG$ is isosceles. Now the author goes on to synthesis; he assumes that $\measuredangle FDG$ is a right angle. Then BD is the altitude and decomposes the triangle into two similar right-angled triangles. Thus, due to similarity

$$\frac{|AD|}{|DB|} = \frac{|AB|}{|BC|}$$

The hypotenuse will be divided in the ratio of the sides.

22.2 Pappos' Rule

The following rules for determining the volume or surface area can already be found in Book VII of the *Collectio* and were attributed to Paul Guldin. Today we know that *P*. Guldin had a copy of the *Collectio* in his library; the naming is probably unjustified.

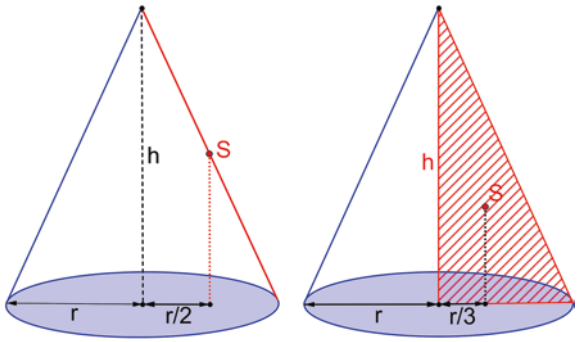

Fig. 22.3 To the rule of Pappus-Guldin

1. If an arc of a curve (without intersection) is rotated around an axis, the envelope surface generated in this way is equal to the product of the arc length ℓ and the path d of the arc center of gravity S during the rotation.

$$M = \ell \cdot \mathrm{d}$$

A cone with height h and radius r of the base circle is considered. For the length of the envelope line, Pythagoras applies $\ell = \sqrt{r^2 + h^2}$. The center of gravity S is in the middle of the arc, its distance from the axis of rotation is $d = \frac{r}{2}$. This results in the envelope surface of the cone (Fig. 22.3)

$$M = \sqrt{r^2 + h^2} \cdot 2\pi \frac{r}{2} = \pi r \sqrt{r^2 + h^2}$$

2. If a surface is rotated around an axis (without overlap), the volume generated is equal to the product of the area $\mathcal{F}$ and the path d of the centroid S during the rotation.

$$V = \mathcal{F} d \quad (2)$$

The volume of a cone (height h, radius of the base circle r) is generated by a right-angled triangle with the catheti r and h. The triangle area is $\mathcal{F} = \frac{1}{2}rh$. The center of gravity S of the triangle lies on a median; from the known part ratio follows for its distance from the axis of rotation $d = \frac{r}{3}$. So it applies to the volume of the cone

$$V = \frac{1}{2}rh \cdot 2\pi \frac{r}{3} = \frac{1}{3}\pi r^2 h$$

Complement Conversely, if the volume is known, the centroid of the generating area can also be determined. In the case of a sphere, the area is a semicircle that rotates around its diameter. For the path of the centroid, (2) (Fig. 22.4)

$$d = \frac{V}{\mathcal{F}} = \frac{\frac{4}{3}\pi r^3}{\frac{1}{2}\pi r^2} = \frac{8}{3}r$$

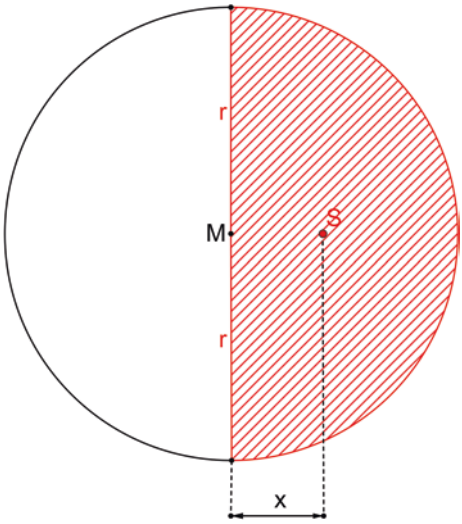

Fig. 22.4 Centroid of a hemisphere

The distance x of the center of gravity from the axis of rotation is therefore

$$x = \frac{d}{2\pi} = \frac{4r}{3\pi}$$

22.3 Pappos' Touching Problem

Pappos simplified Apollonius' Touching Problem. It states:

Given any two points, lines, or circles in a plane, construct a circle of given radius that touches the given points, lines, or circles.

The given objects can be:

1. two points
2. one point and one line
3. one point and one circle
4. two lines
5. one line and one circle
6. two circles

Solution as in Apollonius.

22.4 Pappos' Theorem

Pappos' famous theorem (VII, 139, 143) was one of the first theorems of the later developed field of projective geometry.

If 6 points of an affine plane $\{A, A', B, B', C, C'\}$ are alternately located on 2 lines and these points are connected in pairs ($A \to B', A \to C'$ or $B \to A', B \to C'$ and $C \to A', C \to B'$), then these connecting lines intersect in three collinear points

$$D = AB' \cap A'B \therefore E = AC' \cap A'C \therefore F = BC' \cap B'C$$

In the case of projective completion, the line through D, E, F represents the line at infinity (Fig. 22.5).

Pappus' theorem in the Euclidean plane states:

If: $BC'||B'C$ and $AB'||A'B$, then $AC'||C'A$ also applies (Fig. 22.6).

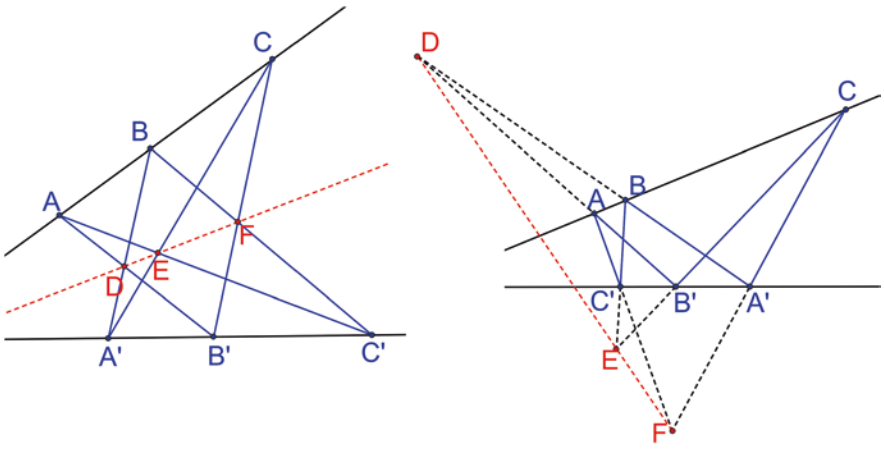

Fig. 22.5 Theorem of Pappos

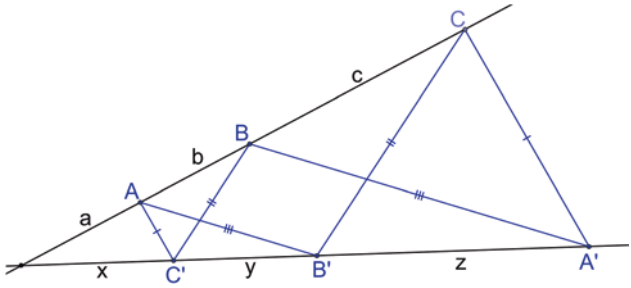

Fig. 22.6 Pappus Theorems in the Euclidean Plane

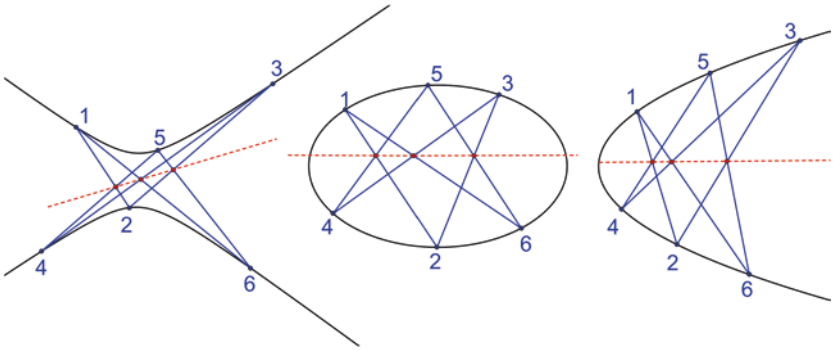

Fig. 22.7 Theorem of Pascal

Proof: According to the similarity of the triangles, the following applies according to the assumption

$$\frac{a+b}{x} = \frac{a+b+c}{x+y}$$

$$\frac{a}{x+y} = \frac{a+b}{x+y+z}$$

Dividing the two equations results in, after transformation

$$\frac{a}{a+b+c} = \frac{x}{x+y+z} \Rightarrow \frac{a}{x} = \frac{a+b+c}{x+y+z}$$

This means: $AC' || C'A$.

If one considers the pair of intersecting lines as a degenerate conic section, then the theorem of Pappus goes over into the theorem of Pascal. Fig. 22.7 shows the hexagram of Pascal for different conic sections.

The theory of conic sections found an important application in the calculation of planetary orbits in the books of Kepler and Newton. The culmination of the theory of conic sections was later the projective geometry in the 19th century.

A proof principle of projective geometry is the so-called principle of duality; it states that a theorem remains correct if one replaces the statement "three points lie on a line" with "three lines intersect in one point" and vice versa. An example of the principle of duality are the theorems of Pascal and Brianchon (Fig. 22.8):

Theorem of *Pascal*: In a hexagon inscribed to an ellipse (or a circle) the points of intersection of diagonals (to two diametrically opposite vertices) are collinear.

Theorem of *Brianchon*: In a hexagon circumscribed to an ellipse (or a circle) the connecting lines of the (diametricaly) opposite vertices are concurrent.

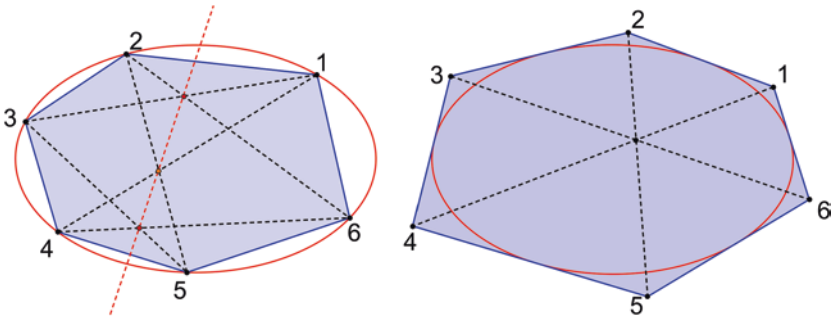

Fig. 22.8 Theorems of Pascal and Brianchon

22.5 The Complete Quadrilateral

The complete quadrilateral consists of the four sides AP, PD, BQ and AQ and the corresponding diagonals AC, BD and PQ. The diagonals are now extended to the intersection with another. This is point S here as the intersection of AC with PQ and T as the intersection of BD with PQ (Fig. 22.9).

Pappos now discovered that the diagonal intersection points R and S divide the diagonal AC externally and internally in the same ratio; i.e. the points $(ARCS)$ are harmonic. This also applies to the other diagonal $(PSQT)$. This means that the two point rows have a central perspective.

The claim is: $\frac{|PS|}{|SQ|} = \frac{|PT|}{|TQ|}$

Modern proof: The Ceva theorem applied to the $\triangle APQ$ with the diagonal through C yields

$$\frac{|PS|}{|SQ|}\frac{|QD|}{|DA|}\frac{|AB|}{|BP|} = 1$$

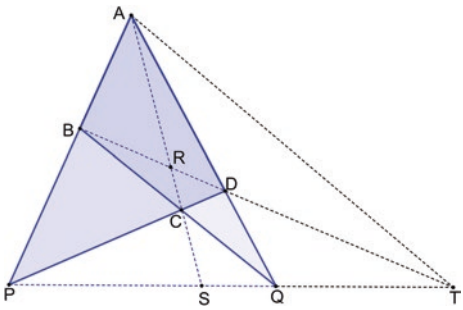

Fig. 22.9 Figure for the quadrilateral

The Menelaos theorem applied to the same triangle with the diagonal BT yields

$$\frac{|PT|}{|TQ|}\frac{|QD|}{|DA|}\frac{|AB|}{|BP|} = 1$$

Immediately setting the two equations equal yields the claim

$$\frac{|PS|}{|SQ|} = \frac{|PT|}{|TQ|}$$

22.6 Harmonic Division

If a line segment AB is divided internally by the point X and externally by the point Y in the same ratio, then the line segment AB is said to be divided harmonically and the four points $(AXBY)$ are harmonic (Fig. 22.10). The designation is explained by the fact that the line segment AB is the harmonic mean of the line segments $|AX|$ and $|AY|$. It holds

$$\frac{|AX|}{|XB|} = \frac{|AY|}{|YB|}$$

With the designation $|AX| = a, |AY| = b$ and $|AB| = m$ follows:

$$\frac{a}{m-a} = \frac{b}{|m-b|} \Rightarrow a(b-m) = b(m-a) \Rightarrow m = \frac{2ab}{a+b}$$

This shows the claim.

Also from Pappos comes the sentence that the polar harmonically divides the secant of a point outside. This construction is found in Apollonius III, 37 in general form for conic sections.

The following construction by Pappos from Book III of the *Collectio* is a special case of harmonic division by means of conic sections (Fig. 22.11). Let $|PT|$ be the diameter of the circle with center M; R a point outside with the tangent $|SR|$. According to the chord-tangent theorem, first

$$|SR|^2 = |PT||RT|$$

The segment $G = |SR|$ is thus the geometric mean of $|RT|$ and $|PR|$. In the right-angled $\triangle MRS$ it holds according to the Pythagorean theorem of Euclid

$$|SR|^2 = |MR||QR|$$

Thus, $|PR||RT| = |MR||QR|$. The line MR is the arithmetic mean A of $|PR|$ and $|RT|$; this follows from

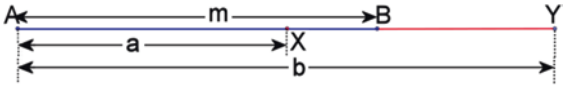

Fig. 22.10 Harmonic division of a line segment

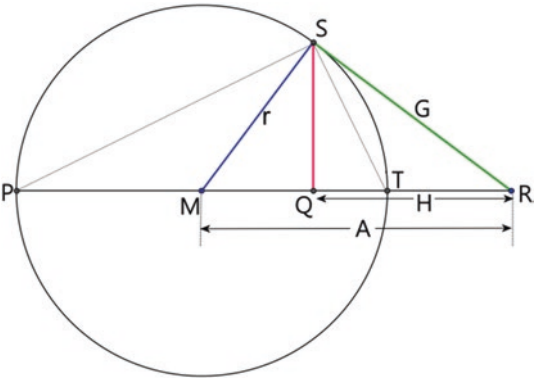

Fig. 22.11 Harmonic division at the circle

$$|MR| = |MT| + |TR| = \frac{1}{2}|PT| + |TR| = \frac{1}{2}\underbrace{(|PT| + |TR|)}_{|PR|} + \frac{1}{2}|TR| = \frac{1}{2}(|PR| + |TR|)$$

QR is thus the harmonic mean H of $|TR|$ and $|PR|$, as shown above, it holds

$$|QR| = \frac{|SR|^2}{|MR|} \Rightarrow |QR| = \frac{G^2}{A} = H$$

The diameter PT is therefore divided by Q or R harmonic.

$$\frac{|PQ|}{|QT|} = \frac{|PR|}{|RT|} \Rightarrow \frac{|PQ|}{|QT|}\frac{|RT|}{|PR|} = 1$$

22.7 The Four-Line Problem

The four-line problem of Pappus, mentioned at the beginning of his book VII, also had great importance in the history of geometry. Pappus took it from Apollonius, but could not solve it in general. The following task is specialized on four lines:

Given are 4 (non-coinciding) lines $a_i (1 \leq i \leq 4)$ and an arbitrary point P. The lines through P, which intersect the given lines a_i at the given angle α_i, generate 4 intersection points Q, R, S and T. To find is the geometric location of the point P, for which it should hold

$$\frac{|PQ|}{|PR|}\frac{|PS|}{|PT|} = k$$

Most often, right angles are chosen for the intersection, so the points Q, R, S and T are the footpoints of the pendiculars dropped from P . R. Descartes dealt with this task in his work *La Géométrie (p. 306),* which was first published in 1637 as an appendix to his famous work *Discours de la Méthode.* The problem comprises with its extensions more than half of Book I and II of the *Geometrie.* Descartes quotes Pappos in it in Latin:

> The problem of the geometric location of 3 or 4 lines, about which he [Apollonios] reports so proudly, and in doing so gives no hint of his predecessors, is of the following nature…

R. Descartes did prove that the sought geometric location is a conic section, which was already suspected by Apollonius. Using the analytical geometry developed by him, he was able to generalize the problem to n lines. Ten years later, I. Newton treated the problem purely geometrically in his Principia (1687) and showed that the geometric location is a conic section (4 lines). In the commentary to his lemma (XIX, cor. II), Newton could not help but take a sideswipe at Descartes. In Latin he wrote:

> What concerns the problem of the four lines, which was raised by Euclid, continued by Apollonius, and solved by me in this proposition without calculus, purely with geometric means, as the ancients required.

The **inversion** is to be shown here:

The case is chosen in which the conic section is a circle through the point P, which lies on the circumference of a cyclic quadrilateral $ABCD$. The four given lines are then the sides of the quadrilateral. In this case it can be shown that the above constant has the value $k = 1$.

The lettering in Fig. 22.12 is chosen. Since each of them contains two perpendicular sides, the quadrilaterals $PQBR$ and $PTDS$ are also cyclic quadrilaterals, since opposite angles complement each other to $2R$.

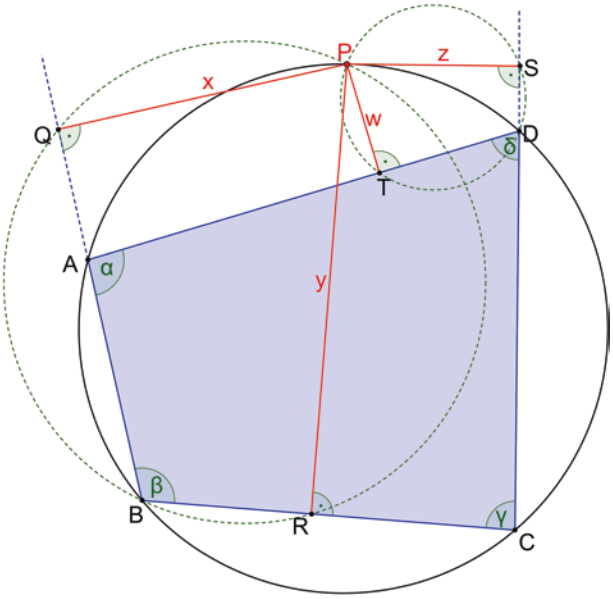

Fig. 22.12 Pappos' 4-line problem

The angle $\measuredangle QPR$ is the supplement of β and thus congruent to δ, since both supplements are in the cyclic quadrilateral $ABCD$. Analogously, the angle $\measuredangle TPS$ is the supplement of $\measuredangle TDS = 180^\circ - \delta$ and thus congruent to δ. Thus, the cyclic quadrilaterals $PQBR$ and $PTDS$ agree in all angles and are therefore similar. For the adjacent sides of the angle δ follows

$$\frac{x}{y} = \frac{w}{z} \Rightarrow \frac{|PQ|}{|PR|}\frac{|PS|}{|PT|} = 1$$

22.8 More Problems of Pappos

Some more topics from the multitude of topics treated by Pappus are to be treated here:

1. **Book IV, 1: Generalization of the Pythagorean Theorem**

Upon the sides AB and BC of the triangle ABC arbitrary parallelograms $DABE$ and $CFGB$ are erected. The extensions of DE and FG intersect in the point H (Fig. 22.13).

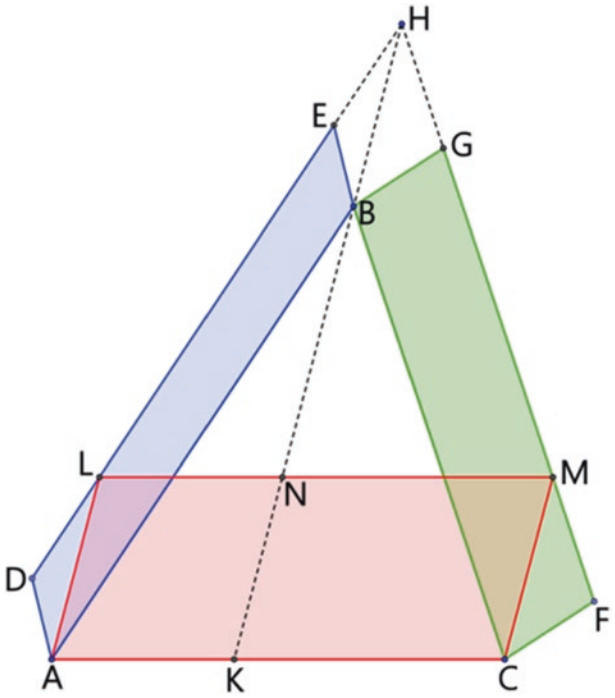

Fig. 22.13 Figure to Pappus IV, 1

Claim: The parallelograms *DABE* and *CFGB* are equal in area to the parallelogram *ACML*, formed by the sides $|AC|$ and $|HB|$ with a base angle $\measuredangle LAC = \measuredangle BAC + \measuredangle DHB$.

Proof: The line *HB* intersects *AC* at point *K*. In *A* and *C* the parallel to *HB* is drawn; the intersection points with *DE* resp. *FG* are *L* resp. *M*. The line *LM* intersects *HB* at *N*. Then *ACML* is a parallelogram with $|AL| = |KN|$. The angle *LAK* is congruent to $\measuredangle BAC + \measuredangle DHB$. The parallelogram *LABH* is equal in area to *DABE* resp. *LAKN*, since each two parallelograms coincide in base and height. Analogously *CFGB* is equal in area to *CMHB* resp. *NKCM*. Thus parallelogram *LACM* is equal in area to the sum of the parallelograms $DABE + CFGB$.

Pappos was proud of his result; he writes: *This result is much more general than the theorem (I, 47) in the Elements about the squares on right-angled triangles.* The theorem also applies if the parallelogram over AC is described outward. In this form, the figure resembles Euclid's theorem (VI, 31), in which similar figures are constructed over the sides of a right-angled triangle.

2. The Hexagon Problem

The last theorem that can be found in Pappos' book VIII, 16 is the problem of inscribing 7 regular, touching hexagons in a circle (Fig. 22.14).

To construct the triangle $\triangle MPQ$ with $\measuredangle MQP = 120^\circ$ and $|MQ| = 2|PQ|$. On the side $|AF| = |MP|$ the points *C*, *E* are marked with $|AC| = \frac{1}{3}|AF|$ and $|CE| = \frac{4}{5}|AC|$. Upon *AC* a circle is constructed to the circumference angle 60° and the tangent of *E* touching the circle at *B*. The distance $|AB| = |QP|$ provides the desired side of the triangle *MPQ*.

3. Pappos chain

In Book IV, 18, Pappos continues the inscribing of circles into the Arbelos figures of Archimedes (Fig. 22.15). The circles of the Pappos chain are numbered as

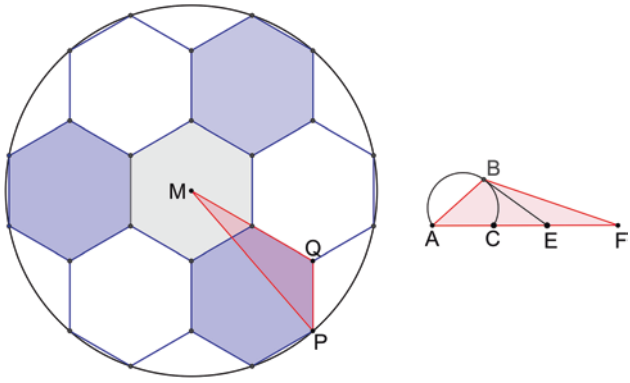

Fig. 22.14 Hexagon-Problem of Pappos

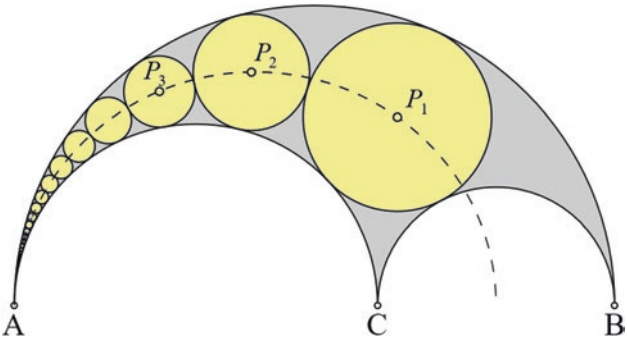

Abb. 22.15 Pappos chain (Wikimedia Commons)

follows: The circle completed over CB Arbelos gets the number 0, the first Arbelos circle gets the number 1, etc. (see figure). The circle centers are designated with P_n, the radii of these circles with r_n. The radii of the two small Arbelos semicircles are $a = \frac{1}{2}|CB| = r_0$; $b = \frac{1}{2}|AC|$. Then the following theorems apply:

- The circle with the number n has the radius $r_n = \frac{ab(a+b)}{n^2a^2+b(a+b)}$
- The center P_n of the circle n has the distance $d = 2nr_n$ from the baseline AB
- The centers of the circles of the Pappos chain lie on an ellipse (dashed). The foci of this ellipse are the centers of the lines $|AB|$ and $|AC|$.
- The points at which the circles of the Pappos chain touch each other lie on a circle.

The figure is often mirrored and a third tangent circle is constructed to each of two touching circles. Continuing this process recursively, one obtains an Apollonius circle filling (according to the tangent problem). In English the corresponding figure is called the *Apollonian Gasket* (Fig. 22.16).

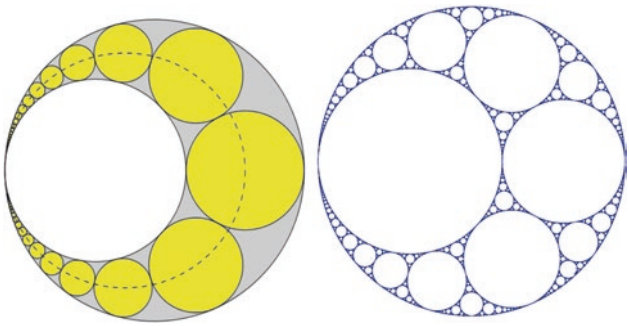

Fig. 22.16 Pappus chain and Apollonius Gasket (Wikimedia Commons)

4. Isoperimetry

At the beginning of Book V, Pappos formulated the famous essay *On the Wisdom of the Bees,* by praising the bees' wisdom to know that of all convex polygons that allow a tesselation (parqueting) of the plane, the honeycombs in the form of matching, regular hexagons have the best surface area given a predetermined circumference (as a measure for the material consumption of wax):

> Although God has given humans, my dear Megethion, the best and most perfect mind for wisdom and mathematics, he has reserved a part of it for some creatures that are not endowed with reason. […] This instinct can be observed in several creatures, most clearly in bees. […] Confident, without doubt, with the task of bringing nectar to the well-educated humans from the gods, they know that it is not appropriate to pour it anywhere on earth or in the wood, but that, by collecting the most beautiful parts of the sweetest flowers growing on earth, they build honeycombs by themselves to store the honey, all of which have a congruent, matching and hexagonal shape. […] The bees therefore know what is useful for them […], that the hexagon is larger than the square and the triangle [given a circumference] and that it can hold more honey with the same amount of material.

Even with the Pythagoreans, the question arose as to whether there are polygons that, given a circumference, have a larger area than others. The first hint of the problem was given by Theon of Alexandria in his commentary on Almagest, in which he quotes K. Ptolemaios:

> Just as with the polygons given a circumference, those with the largest area have the most corners, so the circle in the plane has the largest area and the sphere in space has the largest volume (given a surface area).

As can be seen from Theon's writings, Zenodoros (ca. 200–140 BC) was already interested in isoperimetric figures; however, his writings are lost. After Theon, Zenodoros proved the following theorems:

- Of all the closed curves in the plane of given circumference, the circle has the largest area.
- Of all the closed curves in the plane with the same area, the circle has the smallest circumference.

Pappos also investigates other figures and proves that of all the circular segments (given a circumference), the semi-circle has the largest area; further, that of all the polygons with the same number of corners and the same circumference, the regular n-sided polygon has the largest area. Also: Of all the spatial figures given a surface area, the sphere has the largest volume.

In modern notation, the isoperimetric inequalities can be stated. For any closed curve with circumference U and area A, the following inequality holds:

$$U^2 \geq 4\pi A$$

Equality only occurs for the circle. Analogously, if V is the volume of a three-dimensional region and A is the area of its boundary, then the following inequality holds:

$$A^3 \geq 36\pi V^2$$

Equality only occurs for the sphere.

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Theon of Alexandria

23

23.1 Life and Work

Theon of Alexandria (Θέων ὁ Ἀλεξανδρεύς) (335-405 AD) was a mathematician and astronomer in Alexandria at a time when the Museion had already been destroyed. His lifetime is fixed because he observed the solar eclipse of June 16, 364 and a lunar eclipse in November of the same year in Alexandria, as he reports in his Almagest commentary (Book VI). The Suda states that he lived in Alexandria under Emperor Theodosius I (reign 379–395); furthermore, that he was a member of the Museion. The dating fits the given data; with "Museion" perhaps a successor institution is meant. The chronological table works begun by Ptolemy he continued up to the Consular List of 372. He edited the *Elements* of Euclid for instructional purposes, simplifying some proofs and formulating some suitable lemmas for others.

His main work is the aforementioned commentary on the Almagest of C. Ptolemaios, which is preserved in numerous manuscripts. In the preface he reports that he was urged to do so by the listeners of his lecture. He promises

> … to do better than other commentators who pretend to omit only what is obvious, but go over the true difficulties.

In his commentary on Theon's Almagest, T. Heath writes:

> This commentary is not such as to give us a very high opinion of Theon's mathematical ability; but it is valuable on account of the various historical hints which it gives. We owe to it a useful account of the manner in which the Greeks computed with sexagesimal fractions, illustrated by examples of multiplication, division, and the extraction of roots, as well as by approximation.

Theon produced editions of Euclid's *Elements,* the *Data* and the *Optics.* His version of the Elements was concerned with clarity and freedom from contradiction.

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He inserted additional intermediate steps into more difficult proofs, treated special cases and formulated additional lemmas and corollaries. His edition of the *Elements* can be recognized by the fact that he mentions in his commentary an addition which he made to Proposition [Euclid VI, 33]:

> That sectors of circles in equal circles are related to the angles on which they stand, I have proved in my edition of the Elements at the end of Book VI.

This characteristic allowed it to be determined that all Greek, Arabic and Latin text versions of the Elements which were known by the beginning of the 19th century went back to the text edited by Theon.

In 1808, François Peyrard (1760–1822), then librarian of the Paris École Polytechnique, found an old Greek manuscript in some manuscripts which was recognizable as the text before Theon by the lack of the addition to (VI, 33). This manuscript was stolen from the Vatican Library during the Napoleonic campaign in Italy and brought to Paris by the famous mathematician G. Monge. He used this manuscript for the 1814/1816 edition of a three-volume Greek-Latin-French edition of the Elements which had already been published by him in 1804. This manuscript, now famous as "P" (after Peyrard) or as "Codex Vat. graec. 190", was returned to the Vatican in 1814 after the fall of Napoleon. At the end of the 19th century it served as the main source for the today prevailing Greek-Latin text version of the Elements by J.L. Heiberg, on which all modern editions in living languages are based.

It should be mentioned that in the papyri found in Herculaneum, approximately 120 lines from Book I of Euclid have been found, which do not agree with the Heiberg text to approximately half. The two Arabic translations by *al*-Ḥaǧǧāǧ (end of 8th century) and Ishaq ibn Ḥunayn, later corrected by Thābit ibn Qurra (mid 9th century), also show various deviations. In particular, Wilbur R. Knorr[1] articulated his dissatisfaction with the Heiberg version.

Theon dedicated his commentary to a boy named Epiphanios, who may have been his son. Furthermore, he confirms that his daughter Hypatia contributed to Book III and edited the entire work. Theon and Hypatia are the last (pagan) scientists from Alexandria whose names have been passed down.

23.2 Hypatia of Alexandria

I look up at you and your word in admiration,
Like the constellation of Virgo that shines in the sky.
For all your actions and thoughts strive heavenward,
Hypatia, you noble one, sweet voice of reason,
Learned and immaculate star of wisdom![2]

[1] Knorr W.R.: The Wrong Text of Euclid: On Heiberg's Text and its Alternatives*, Centaurus 1996, Vol. 38, pp. 208–276.

[2] Palladas, Anthologia Graeca IX, 400.

Fig. 23.1 Hypatia (Wikimedia Commons)

Hypatia (Ὑπατία) of Alexandria (370–415) was the first mathematician whose name was recorded. Since girls were not allowed to attend the *Gymnasion*, she was given a comprehensive education by her father. Her father was Theon of Alexandria, a Platonist philosopher and astronomer at the successor institution of the Museion; the original Museion had been destroyed in 270 AD.

Fig. 23.1 shows a portrait of Hypatia by Jules M. Gaspard. She therefore worked at the Serapeion, which consisted of the Serapis temple founded by Arsinoe II and the remaining library of the Museion. When Emperor Theodosius ordered all pagan temples to be destroyed in 392 AD, the rest of the library was also burned. At the Serapeion, Hypatia taught not only the philosophy of the Platonic school, but also mathematics and astronomy. A student of Hypatia wrote about her:

> In her philosopher's robe, she walked through the city center and spoke publicly to all who wanted to listen about the teachings of Plato or Aristotle. […] The magistrates used to consult her first for advice on state affairs.

The contemporary Sokrates Scholasticus (ca. 380–439) from Constantinople wrote in his *Historia Ecclesiastica:*

> There is a woman living in Alexandria, by the name of Hypatia, daughter of the philosopher Theon, who made such progress in literature and science that she soon surpassed all the philosophers of her time. After completing her studies of Plato and Plotinus, she explained the principles of philosophy to her listeners, many of whom came from far away to be instructed by her.

She also constructed an astrolabe and a hydroscope at the request of *Synesius* of Cyrene, the later bishop of Ptolemaïs (today in Libya). In his letters, Synesius

Fig. 23.2 The murder of Hypatia, AKG182994 Copyright/akg-images

called her *mother, sister, and revered teacher.* In one letter, he asks her for a judgment on two books he wishes to publish

> If you think I should publish my books, I will dedicate them to all the rhetoricians and philosophers. The former will be pleased, the latter will find them useful; provided that you approve of the project, for you are truly capable of passing judgment. If it is not worth being heard by Greek ears, then, like Aristotle, you value truth above friendship; a profound darkness will overshadow the project and humanity will never learn of it.

The Suda writes that she is attributed three works, namely a detailed commentary on the arithmetic of Diophantus, the conic sections of Apollonius, and the astronomical *Kanon.* It is also certain that she contributed to Book III of Theon's Almagest commentary, as her father writes at the beginning of the book.

Since its founding, Alexandria had become a melting pot for various religious denominations. From a Christian perspective, Hellenism developed into a pagan religion with a multitude of gods in Egypt. Since the Ptolemaic Empire also included Syria, many Jews also lived in Alexandria. After the division of the Roman Empire, Alexandria fell to the East (= Byzantium). Since the (Eastern) Roman Emperor Theodosius had made Christianity the state religion in 380 AD, Christianity was the official confession. Religious zealots repeatedly incited violent confrontations, among other things they looted the Serapeion in 391 AD. One of these violent outbursts finally led to the death of Hypatia in March 415, who was considered the last representative of the pagan faith (Fig. 23.2).

Tradition has it that Cyril, who was bishop of Alexandria since 412, incited a number of Christian zealots to murder Hypatia. He pretended that Hypatia had an evil influence on the Roman prefect Orestes. Since he prevailed against the resistance of Nestorius at the Council of Ephesus (431) and established the motherhood of Mary, he was even canonized. With the death of Hypatia, the Neoplatonic school of Alexandria died out. In 529, the (late) Academy in Athens was also closed by Justinian. This meant that a large part of the scientific knowledge of antiquity was lost forever. A certain part of the works survived only because it was preserved and transmitted by Byzantine or Arab-Islamic scholars or because a Latin translation already existed.

Tertullian (160–220 AD), the first Christian author writing in Latin, remarks in *De Praescriptione Haereticorum* VII (in reference to Acts of Apostles, 5,12):

> What indeed has the Athens to do with Jerusalem? What concord is there between the Academy and the Church? What between heretics and Christians? Our instruction comes from the *porch of Solomon*, which had taught us that *we must seek the Lord in simplicity of our heart*. Away with all attempts to produce a mottled Christianity of Stoic, Platonic, and dialectic composition! We want no curious disputation after possessing Christ Jesus, no inquisition after enjoying the gospel!

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Proclus Diadochus

24

Proclus (411–485 AD) was born in Byzantium. He was supposed to, like his father Patricius, become a jurist. He studied rhetoric and jurisprudence in Xanthus (Lycia) and later in Alexandria. At the request of the Roman governor Theodorus, the young Proclus accompanied the scholar *Leonas* of Isauria to Byzantium. The encounter at the court of Byzantium with the empress Eudocia and her father Leontius aroused in him the desire to dedicate himself to philosophy. He first returned to Alexandria to study logic and mathematics there. With these knowledge he took up the study at the rebuilt Academy in Athens at the age of 19. As his talent was soon recognized, the head of the Academy *Syrianus* took him into his household. Syrianus persuaded his predecessor Plutarch to train him specifically in Aristotle and parts of the Platonic work. After the sudden death of Syrianus in 437, Proclus became his successor and was therefore given the nickname *Diadochus* (= successor) and took over his Platonic philosophy lectures. According to the report of his student Marinus, Proclus lived an ascetic life and studied all his life. In Vita Procli (§13) Marinos reports:Working day and night with tireless discipline and care, and writing down what was said in a comprehensive yet discriminating manner, Proclus made such progress in a short time that, when he was still in his twenty-eighth year, he wrote a great many treatises, which were elegant and teeming with knowledge, especially the one on the Timaeus. Other well-known students were Isidore of Miletus, Ammonius and Zenodotus.

Proklos wrote numerous comments on the Platonic works, but only a fraction of them have been preserved. His interpretations of Homer and Hesiod are also lost. However, his commentary on the first book of Euclid is completely preserved. This is a lucky chance for the History of Mathematics, because the commentary quotes much of Eudemos' *Catalogue of Mathematicians*, which the latter wrote in 370 BC on behalf of Aristotle. The Eudemian Summary is a valuable source of information about earlier mathematicians. In his commentary on the Timaeus, Proklos expresses dissatisfaction with Ptolemy's theory of the planets.

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An important quotation from the preface to Book I of the Euclid commentary is used by J. Kepler[1] in the prefaces to books III and IV of his *Harmonices Mundi*:

> From what we have said it is clear that mathematical science makes a contribution of the greatest importance to philosophy and to its particular branches, which we must also mention... It proves that numbers reflect the properties of beings above being and in the objects studied by the understanding reveals the powers of the intellectual figures. Thus Plato teaches us many wonderful doctrines about the gods by means of mathematical forms and the philosophy of the Pythagoreans clothes its secret theological teaching in such draperies.

He continues:

> Mathematics also makes contributions of the very greatest value to physical science. It reveals the orderliness of the ratios according to which the universe is constructed and the proportion that binds things together in the cosmos, making, as the Timaeus somewhere says, divergent and warring factors into friends and sympathetic companions... And it discovers, furthermore, the numbers applicable to all generated things and to their periods of activity and of return to their starting-points, by which it is possible to calculate the times of fruitfulness or the reverse for each of them.

Proclus also criticizes the formulation of Euclid's Parallel Postulate, which does not appear to him to be evident. He writes

> … just as is also assumed in the postulates what is immediately evident and causes no difficulty to our untrained thought, while in the axioms we seek to find what is easy to procure and to determine.

He points out that already Geminos of Rhodes had observed that there are certain curves and straight lines which approach one another to infinity without, however, intersecting, like the asymptotes of a hyperbola.

Also interesting are his remarks on the gradations of infinite quantities. He observes that there are infinitely many ways of halving a circle by means of a diameter. But the two halves of the circle can be combined in two different ways; thus the latter quantity is greater than that of the circle-halvings.

Proclus himself has a method that allows ellipses to be constructed point by point (Fig. 24.1).

If you are looking for an ellipse with the semi-axes (a, b), you construct two concentric circles K_1, K_2 with these radii. If you draw the radius vector at a given angle, you will obtain the intersection points P and Q with both circles. The perpendicular from Q to the horizontal axis and the parallel to the horizontal axis through P intersect in the ellipse point R. If P passes through the circle K_2 and Q passes through the circle K_1, the construction will provide the desired ellipse

[1] Kepler J., Kraft F. (eds.): What holds the world together in its innermost, reprint Marix 1923, p. 178.

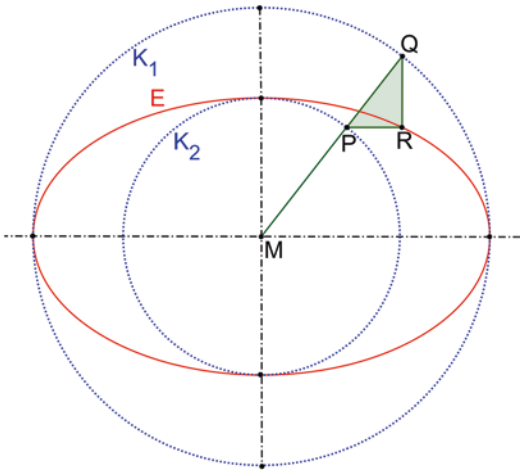

Fig. 24.1 Pointwise construction of an ellipse

E. This image of the circle K_1 on the ellipse is a stretching in the y direction: $y' = \frac{b}{a}y \Rightarrow y = \frac{a}{b}y'$ with the stretch factor $\frac{b}{a}(b < a)$. The circle equation K_1 is

$$x^2 + y^2 = a^2 \Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

The application of the stretch provides the desired ellipse with the semi-axes (a, b).

$$\left(\frac{x}{a}\right)^2 + \left(\frac{\frac{a}{b}y'}{a}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

24.1 The Mathematician's Catalogue of Eudemus

In the preface to Book II of his commentary on Euclid, Proclus[2] gives a section of the history of mathematics of Eudemus again. Eudemus had written four books on the history of *mathematics* and *astronomy* on behalf of Aristotle in 334. The report contains numerous names of mathematicians of the Academy, which are otherwise not documented in other sources. The report is therefore also called the *mathematician-directory*.

> As in the case of the Phoenicians, the beginnings of a precise knowledge of numbers arose from trade and commerce, so too in the case of the Egyptians, geometry was created for the aforementioned reason. But it was Thales who, after coming to Egypt, first transplanted this science to Greece and made many discoveries himself; he laid the foundations for many others. His approach was partly of a more general nature, partly more oriented

[2] Proclus Diadochus, Steck M. (ed.): Euclid Commentary, German Academy of Natural Scientists Halle 1945, p. 211–214.

towards final causes. After him, it was Mamertius, the brother of the poet Stesichorus, who, according to tradition, occupied himself with the study of geometry, and Hippias of Elis reported that he achieved fame because of geometry; they were followed by Pythagoras, who incorporated their scientific activity into the system of higher education. His investigations were directed towards their highest principles, and his theoretical research moved freely in the realm of pure thought, uninfluenced by material factors. He was also the one who created the doctrine of proportions and the representation of the five world bodies. After these, Anaxagoras of Clazomenae dealt with many problems of geometry, and then, a little younger than Anaxagoras, Oinopides of Chios. Plato mentions both of them in the Rivals as having become famous because of their mathematical knowledge. After these were Hippocrates of Chios, the inventor of the quadrature of the lune, and Theodorus of Cyrene, notable exponents of the field. For Hippocrates appears in history as the first author of an elementary book.

This was followed by Plato, whose diligent study is responsible for the fact that the other mathematical disciplines and especially geometry made the greatest progress. It is obvious that he has completely permeated his writings with mathematical thought and everywhere tries to evoke wonder in those interested in philosophy. At this time, Leodamas of Thasos, Archytas of Tarentum, and Theaetetus of Athens also lived, from whom the doctrines were increased and brought into a system more in line with scientific requirements. Younger than Leodamas is Neocles and his pupil Leon, who considerably extended the knowledge of their predecessors, so that Leon was able to write a well-founded elementary book on this subject in terms of richness and usefulness of the evidence, and also give precise specifications for when the solution to a given task is possible or not. Eudoxus of Cnidus, a little younger than Leon and a friend of Plato's pupils, was the first to increase the number of so-called general doctrines, adding three more to the three proportions and further developing the doctrine of the division of the line, which Plato had laid down, in several sentences, using the analytical method.

Amyntas of Heraclea, a friend of Plato, Menaichmos, a student of Eudoxus, who was also contemporary with Plato, and his brother Deinostratus completed the whole system of geometry more perfectly. Theudius of Magnesia was also outstanding in the field of mathematics and other sciences. For he reduced the elementary doctrine to a system and gave many definitions a general form. Athenaeus of Cyzicus, who belonged to the same time, made a name for himself by his achievements in the other mathematical disciplines and especially in geometry. All of these lived together in the Academy and pursued their research together. Hermotimus of Colophon developed the results of Eudoxus and Theaetetus further, made a significant contribution to the elements and wrote one on (geometric) loci. Philip of Medma, Plato's student and stimulated by him to study mathematics, pursued his research according to Plato's instructions and only set himself such tasks as he hoped would promote Platonic philosophy. The historians trace the development of this science back to him.

Proklos(?) supplements the Eudemos report:

Not much younger than this is Euklid, who put together the Elements, brought together many results of Eudoxos, completed many of Theaitetos, and reduced the less stringent proofs of his predecessors into irrefutable form. He lived in the time of the first Ptolemy. For Archimedes, who lived after the first Ptolemy, mentions Euklid and also relates in fact that Ptolemy once asked him whether there might not be a shorter way to geometry than the Elements' doctrine. But he replied that there is no royal road to geometry. He is younger than Plato's pupil, but older than Eratosthenes and Archimedes; for these are contemporaries, as Eratosthenes says somewhere. He belonged to the Platonic school and was familiar with this philosophy, which is why he also set the representation of the so-called Platonic solids as the goal of the entire elementary doctrine. A large number of

other mathematical writings also originate from him, all characterized by admirable accuracy and scientific speculation.

This part about Euclid cannot be written by Eudemus, since Euclid lived after him. This was already noticed by B. Tannery in his work *La géométrie greque* (1887); he suspected *Geminus of Rhodes* as the author. This is contradicted by T. Heath (I, p. 109), who attributes the authorship to a later writer. The problem is still open today; it is also conceivable that Proclus continued the report himself. Part of Eudemus' *History of Mathematics* has also been preserved in Simplicius' commentary on Aristotle's Physics. This part is about the quadrature of the *Hippocrates* from Chios.

24.2 More Important Quotes

1. To the parallel problem (p. 287)

Poseidonios but says, parallel are the lines which in a plane neither approach nor recede from each other, but all have the same perpendiculars drawn from the points of one to the other. Those which form always smaller perpendiculars run together; for the perpendicular can determine the heights of places and the distances of the lines. Therefore, if the perpendiculars are equal, the distances of the lines are also equal; but if they become greater or smaller, the distance also diminishes, and they run together on the side where the perpendiculars become shorter.

Asymptotes which lie in a plane always preserve the same distance from each other, the others always reduce the distance more and more in relation to the line, like the hyperbola and the conchoid. For these always remain as asymptotes with a continual reduction of the distance; they indeed approach each other, but never entirely. Of the lines which always preserve the same distance, the lines which never reduce their distance and lie in a plane are parallel.

2. Attempted proof by Ptolemy (p. 422)

After Ptolemy had previously proven this [sum of the adjacent angles of parallels equal to 2R] and had come to the present proposition, he wanted to add a special finesse and show that if a line intersects two lines in such a way that the angles on the same side are smaller than 2 right angles, the lines are not only not asymptotes, as was shown, but their intersection also occurs *on* the side where the angles are smaller than 2R, not where they are larger. Let the two lines AB and CD and EFGH intersect in such a way that the angles AFG and CGF together are smaller than 2R. Then the remaining angles will be larger than 2R. But if they intersect, they will intersect either on the side of AC or on the side of BD. They should now intersect on the side of BD at the point K. Since the angles AFG and CGF together are now smaller than 2R, and the angles AFG+BFG = 2R, then after subtracting the common angle AFG, the angle CGF will be smaller than the angle BFG. So the exterior angle of the triangle is smaller than the opposite interior angle, which is impossible. But they intersect. They will therefore intersect on the other side, where the angles are smaller than 2R. So far Ptolemy (Fig. 24.2).

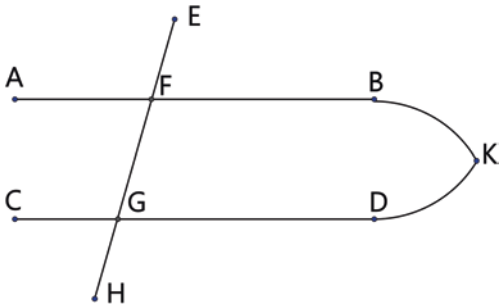

Fig. 24.2 Ptolemy's proof attempt

3. Infinite sets (p. 276)

> But if there is only one diameter and, on the other hand, two semicircles, and an infinite number of diameters go through the center, then the result is a doubling of the infinite number. For some people raise this difficulty when they divide magnitudes that are carried to infinity. But we observe that the magnitude is indeed carried to infinity, but not to an infinite number. For the latter only really causes infinity, the former only in potency, and the latter conditions the nature of the infinite, the former only its origin. At the same time with one diameter there are also two semicircles, and the diameters will never be infinite, even if they are taken to infinity, so that there will never be a doubling of the infinite, but the resulting doublings will always be doublings of finite numbers.

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Roman Mathematics

25

Since the Romans built enormous aqueducts, fortresses, port facilities, border walls, bridges throughout the empire – from Scotland to the Caspian Sea – they must have had sound knowledge of logistics, engineering and architecture (cf. Fig. 25.1). However, no creative mathematician is known to have been active during Roman times. Mathematics was only an auxiliary science for surveyors *(agrimensores),* engineers and architects among the Romans. This view can be read, for example, in Cicero:

> Everyone knows how dark the field of mathematics is, how remote, complicated and pedantic the science is that they deal with. Nevertheless, there have been many so-called mathematicians, that one gets the impression that hardly anyone who has dealt with mathematics seriously has failed [De oratore I, 3,10].

> The Greeks highly esteemed geometry, and so no one was more respected than the mathematicians. Geometry was held in the highest regard by the Greeks. Therefore, nothing was more important to the mathematicians than geometry; but we [Romans] have limited the scope of this science to the utility it brings us in measuring and calculating [Tusc. Disp. I,2,5].

> If this science has any value at all, it is only that it sharpens the minds of young people a little and provides them, as it were, with an incentive, so that they can more easily learn what is more important [De Re Pub I,18].

Cicero was the first Roman author to mention Euclid (*De oratore* III, 132). The author Claudius Aelianus[1] (*Aelian*) (c. 220 AD) mocks:

> It seems that all spiders ... know geometry by nature. They sit in the center of their web and keep the woven circles with great accuracy although they don't know anything about Euclid.

[1] Aelianus Claudius: De natura animalium, Edition Ambrosio Didot, Paris 1858, Buch VI, 57.

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D. Herrmann, *Ancient Mathematics*, https://doi.org/10.1007/978-3-662-66494-0_25

Fig. 25.1 Forum Romanum at the time of Caesar (Wikimedia Commons)

The attitude of Seneca, Nero's teacher, is mixed; he writes in his letters:

> That the sun is large, the philosopher will prove to you; how large the sun is, the mathematician, who comes to a result through a kind of application and exercise; but to obtain this, he must have certain principles [Epist. 88.27].

> Should I stick to the dust [in which the figures are drawn]? Have I already moved so far away from the healthy maxim: Be sparing with your time! That [geometry] I should know? And what should I leave out for that? [Epist. 88.42].

J.L. Heiberg[2] commented on the above Cicero quote polemically:

> The arch-dilettante Cicero once praises his countrymen that they, thank God, are not like those Greeks, but limit the study of mathematics and the like to the practically applicable and useful. Therefore, the Romans have not achieved anything independent in the areas concerned; what they needed, they borrowed from the Greeks.

It should be mentioned that mathematics was "pagan heritage" that was denounced during the Christianization. For example, in the *Codex Justiani*[3] (Book IX, $ 18):

> **Of malefactors, mathematicians and the like** (*De maleficis et mathematicis et ceteris similibus*).
>
> (§18.2) The art of teaching and practicing geometry is a public duty. The art of mathematics, however, is condemned and entirely forbidden.

[2] Heiberg J. L.: Natural Sciences and Mathematics in Antiquity, p. 73, Teubner Leipzig 1912.

[3] www.opera-platonis.de/CI/

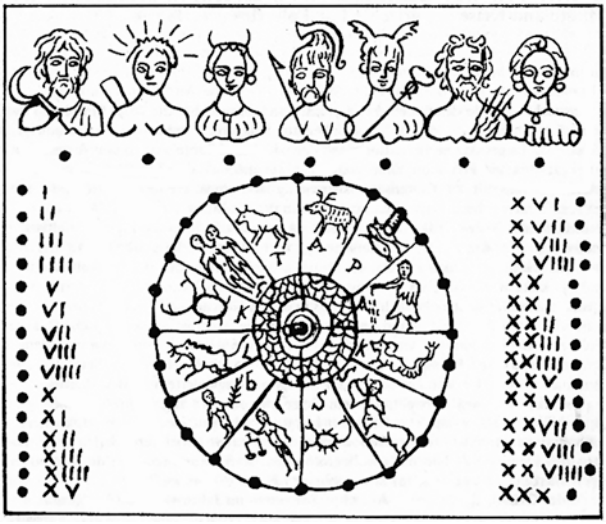


Fig. 25.2 Plug-in Roman Calender, Lexikon der Technik (Leipzig, 1980)

(§18.4) It is punishable and rightly pursued by the strictest jurisdiction, the sorcery of those who either act against the well-being of mankind or have led a chaste mind to lust.

(§18.5) No one should consult a sacrificial viewer (*haruspex*) or mathematician, no one should consult a fortune-teller (*hariolus*) …

25.1 Calculating with Roman Numerals

The first 30 Roman numbers are shown on the plug-in calendar found in Rome *(parapegma)* (around 350 AD). In the middle you can see the twelve zodiac signs, at the top the seven gods (from left Saturn, Sol, Luna, Mars, Merkur, Jupiter and Venus), which provide the names of the weekdays. On the left and right are the slots for 30 days of the Roman month (Fig. 25.2).

The first fundamental study of Roman arithmetic dates back to G. Friedlein in his work *Number Signs*[4]. Preceded by an article on Roman numerals by M. Cantor[5] in his work on cultural life. Friedlein's book had little international resonance.

[4] Friedlein G.: The number signs and the elementary arithmetic of the Greeks and Romans, Deichert Erlangen 1869.

[5] Cantor M.: Mathematical Contributions to the Cultural Life of Peoples, Halle 1863, Reprint Olms 1964, pp. 155–167.

The international history of mathematics made the topic of Roman arithmetic relatively late. A newer American principle article [6] by D. Mahrer and J. Makowski begins with the words:

> Roman arithmetic is an ongoing problematic issue, both for ancient philologists and for mathematicians. Scholars worldwide report the difficulties that arise from the [purely] verbal description, both in geometric figures and in the execution of written calculation.

A good description of how Roman fractional arithmetic could have worked in particular is provided by W.F. Anderson [7] in his principle article (1956).

The unreadability of Roman numerals and the lack of a place value system baffled some authors. L. Motz and J. Weaver write: *Is there anyone who can carry out any calculation with DCCCLXXXVIII?* F. Cajori[8] states:

> While older number systems are used to fix results, the wonderful power of Indian notation helps to carry out the calculation itself. To test this, you should multiply 723 by 364 and carry out the product DCCXXIII times CCCLXIV in Roman notation. The latter notation provides little to no assistance. So the Romans are forced to use an abacus for such calculations.

J. Tropfke (1930) writes about Roman arithmetic:

> The long persistence of this highly unskilled system, in which difficult calculations of Roman engineers and surveyors only became more difficult and unreadable, is testimony to the low scientific-mathematical contribution of the Romans.

In contrast to Egyptian and Mesopotamian sources, we do not have a work that provides a precise description of Roman arithmetic. Even of the popular finger arithmetic there are records only with *Beda* Venerabilis. Here, the multiplication tables up to 5 had to be learned by heart, larger one-digit numbers were multiplied according to the formula:

$$a \cdot b = 10(a + b - 10) + (10 - a)(10 - b); a > 5; b < 10$$

Example is: $9 \cdot 7 = 10(9 + 7 - 10) + (10 - 9)(10 - 7) = 60 + 3 = 63$

The transposed formula was probably only used later:

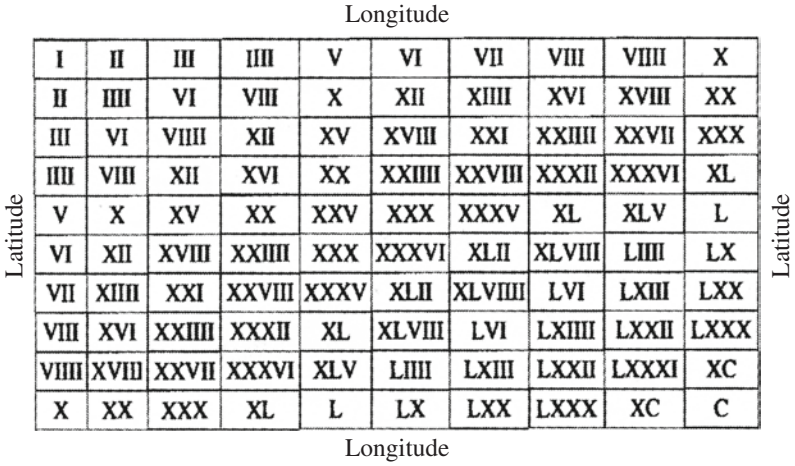
$$a \cdot b = 10(a + b - 10) - (a - 10)(10 - b); a \geq 10; b < 10$$

Pattern is: $14 \cdot 7 = 10(14 + 7 - 10) - (14 - 10)(10 - 7) = 110 - 12 = 98$

[6] Maher D. W., Makowski J. F.: Literary evidence for Roman arithmetic with fractions, p. 376–399, Classical Philology 96 (2001).

[7] Anderson W. F.: Arithmetical computations in Roman numerals, p. 154 ff., Classical Philology 3 (1956).

[8] Cajori F.: A History of Elementary Mathematics, p. 11, New York 1950.



I	II	III	IIII	V	VI	VII	VIII	VIIII	X
II	IIII	VI	VIII	X	XII	XIIII	XVI	XVIII	XX
III	VI	VIIII	XII	XV	XVIII	XXI	XXIIII	XXVII	XXX
IIII	VIII	XII	XVI	XX	XXIIII	XXVIII	XXXII	XXXVI	XL
V	X	XV	XX	XXV	XXX	XXXV	XL	XLV	L
VI	XII	XVIII	XXIIII	XXX	XXXVI	XLII	XLVIII	LIIII	LX
VII	XIIII	XXI	XXVIII	XXXV	XLII	XLVIIII	LVI	LXIII	LXX
VIII	XVI	XXIIII	XXXII	XL	XLVIII	LVI	LXIIII	LXXII	LXXX
VIIII	XVIII	XXVII	XXXVI	XLV	LIIII	LXIII	LXXII	LXXXI	XC
X	XX	XXX	XL	L	LX	LXX	LXXX	XC	C

Fig. 25.3 Times table from a French Boethius manuscript (Guillaumin 1995)

The first description of a *multiplication table* can be found in Boethius, however it is meant as an array of multiples of 1 to 10. Fig. 25.3 shows a multiplication table from a French Boethius[9] -manuscript.

The *first* writing "Assis Distributio", which represents all fractions named, originates from L. Volusius *Maecianus* from the year 146 BC. The first work of tables for multiplication and division of Roman numbers is not provided until the cleric *Victorius* from Aquitaine (5th century) in his work *Calculus*.[10]. The text was later commented on in detail by Abbo of Fleury, so that its content can be largely reconstructed. The date of Victorius' death is known: In 457, he wrote a work on the determination of the Easter date *(Cursus paschalis) on behalf of the Roman archdeacon and later Pope Hilarius.* Fig. 25.4 shows two columns of Victorius' division tables, here the divisions by {12; 13}. If a dividend does not appear in the table, it must be decomposed into terms that are tabulated. As an example, the (integer) division $1027 \div 13$ should serve:

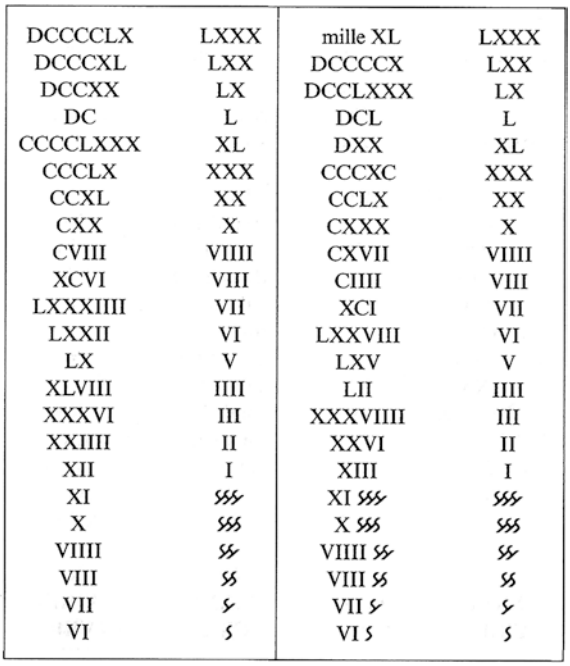
$$MXXVII \div XIII = (DCCCCX + CIIII + XIII) \div XIII = LXX + VIII + I = LXXVIIII$$

As another pattern, the doubling table is shown here (Fig. 25.5):

Reading old manuscripts with breaks is difficult, as some number representations have changed over time (up to the introduction of Indian numerals), as can be seen, for example, in the table of powers of ten:

[9] Guillaumin J.-Y. (ed.): Boèce, Institution arithmétique, Les Belles Lettres, Paris 1995, p. 54.

[10] *Victorii calculus ex codice Vaticano editus,* Friedlein G. (Hrsg.): Bullettino di bibliografia e di storia delle scienze matematiche e fisiche. Band 4 (1871), S. 443–463.



DCCCCLX	LXXX	mille XL	LXXX
DCCCXL	LXX	DCCCCX	LXX
DCCXX	LX	DCCLXXX	LX
DC	L	DCL	L
CCCCLXXX	XL	DXX	XL
CCCLX	XXX	CCCXC	XXX
CCXL	XX	CCLX	XX
CXX	X	CXXX	X
CVIII	VIIII	CXVII	VIIII
XCVI	VIII	CIIII	VIII
LXXXIIII	VII	XCI	VII
LXXII	VI	LXXVIII	VI
LX	V	LXV	V
XLVIII	IIII	LII	IIII
XXXVI	III	XXXVIIII	III
XXIIII	II	XXVI	II
XII	I	XIII	I
XI	ſſſ	XI ſſſ	ſſſ
X	ſſ	X ſſ	ſſ
VIIII	ſ	VIIII ſ	ſ
VIII	ſ	VIII ſ	ſ
VII	ſ	VII ſ	ſ
VI	ſ	VI ſ	ſ

Fig. 25.4 Division by {12; 13} according to Victorius

1000	(I); ↀ; ∞
10.000	((I)); ⓘ
100.000	(((I))); ⓒ
1.000.000	$\lvert\bar{X}\rvert$

The early forms of the number symbols for 1000 or 100,000 can be found on the *Columna Rostrata* (the red column on the right in Fig. 25.1) on the *Forum Romanum* near the speaker's platform *(rostrum),* in which (before Caesar's renovation) the prow rams of the captured ships were built into the wall. The column contains a tablet (Fig. 25.6) with a list of the war booty that the Romans captured in the sea battle of Mylae (260 BC) from the Carthaginians. The sign for 100,000 (ⓒ) after the word "AES" *(copper)* can be found 34 times. The unit of currency was the *As;* this originally corresponded to the value of a copper bar weighing 1 pound. Pliny the Elder confirms that there was no sign for numbers greater than 100,000 at that time:

> There was no number greater than 100,000 among the ancients; therefore, these are also multiplied today, so that they are said to be ten times ten thousand or more often.
>
> Later it became customary to indicate multiples of 1000 by a line above; thus, $\overline{XXX}$ *MM* reads as 32,000. Multiples of 10,000 were marked by a box below. Also, the position

Bis	distas	[ſſſ]	id est	assis et bisse	[I ſſ]
Bis	iabus	[ſſſ]	id est	assis et distas	[I ſſſ]
Bis	assis	[I]	id est	dipondius	[II]
Bis	bini	[II]	id est	quaterni	[IIII]
Bis	terni	[III]	id est	seni	[VI]
Bis	quaterni	[IIII]	id est	octeni	[VIII]
Bis	quini	[V]	id est	deni	[X]
Bis	seni	[VI]	id est	decus dipondius	[XII]
Bis	septeni	[VII]	id est	decus quartus	[XIIII]
Bis	octus	[VIII]	id est	decus sextus	[XVI]
Bis	nonus	[VIIII]	id est	decus octus	[XVIII]
Bis	deni	[X]	id est	viceni	[XX]
Bis	vigeni	[XX]	id est	quadrageni	[XL]
Bis	trigeni	[XXX]	id est	sexageni	[LX]
Bis	quadrageni	[XL]	id est	octogeni	[LXXX]
Bis	quinquai	[L]	id est	cean	[C]
Bis	sexai	[LX]	id est	ceanbiae	[CXX]
Bis	septai	[LXX]	id est	cean quadrai	[CXL]
Bis	octai	[LXXX]	id est	cean sexai	[CLX]
Bis	nonai	[XC]	id est	cean octai	[CLXXX]
Bis	cean	[C]	id est	ducen	[CC]
Bis	ducen	[CC]	id est	quadricen	[CCCC]
Bis	tricen	[CCC]	id est	sexacen	[DC]
Bis	quadricen	[CCCC]	id est	octicen	[DCCC]
Bis	quinquien	[D]	id est	chile	[I milia]

Fig. 25.5 Doubling of Roman numbers according to Victorius

Fig. 25.6 Tablet of war booty on the Columna Rostrata (victory column) with redrawn (Wikimedia Commons)

of the signs was not standardized because the subtractive principle was not generally observed. So there are the spellings $(XC, LXXXX)$ for 90 or $(CCM, DCCC)$ for 800.

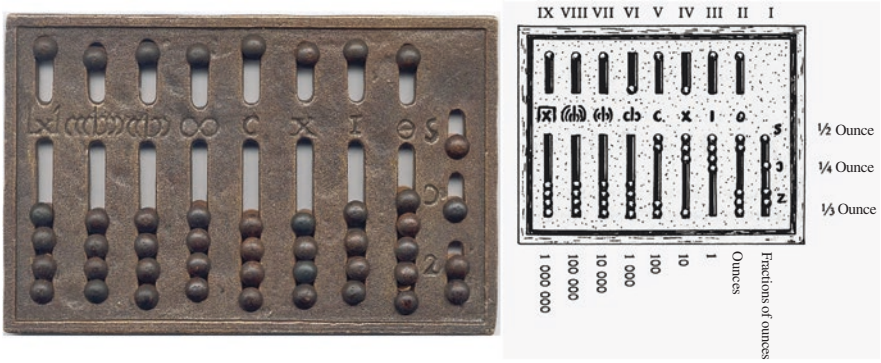


Fig. 25.7 Roman abacus with notation (Wikimedia Commons)

Another difficulty still arises with Roman fractions, namely that only certain, like multiples of 1/12, have a name; all other fractions had to be represented as multiples. Another, less used, way to name a fraction is the word *pars,* which connects the numerator and denominator; *partes assis* is the word for a fraction. So they mean

- Viginti *pars* undesexagesima = 20/59,
- Diei *pars* MDCXXIII = 1/1623 Tag.

You can also find number symbols for ounces on a Roman abacus, i.e. fractions with a denominator of 12 (Fig. 25.7). Smaller fractions could be indicated as multiples of ounces, e.g. $\frac{1}{36}$ as 1/3 ounce *(triens unciae);* since $\frac{1}{72}$ was also called *sextula*, the name *binae sextulae* $\left(2 \times \frac{1}{72}\right)$ was also possible.

Fig. 25.8 contains the names of all fractions; it shows clearly how the fractional symbols have changed over three centuries (from Tropfke[11]) (Fig. 25.8):Table of named Roman fractions according to Tropfke

Roman fractional symbols rarely appear in literature. Smith[12] has drawn up a list of frequencies with which the fractional symbols appear: Varro (12), Maecianus (14), Isidor (8).

According to H. Hankel[13] the following term is to be calculated using Roman fractions:

$$\left(1\frac{1}{3}\right)^2 \div \left(1\frac{1}{4}\right)^2$$

[11] Tropfke J., K. Vogel, K. Reich, H. Gericke (eds.): History of Elementary Mathematics Volume 1, De Gruyter New York 1980, p. 105.

[12] Smith D.E.: History of Mathematics, Volume II, Dover 1958, p. 215.

[13] Hankel H.: Zur Geschichte der Mathematik im Altertum und im Mittelalter, Reprint Bibliolife 2010, p. 61.

Bruchteil des As	Name	Erklärung	Zeichen Volusius Maecianus	Victorius
$\frac{1}{2}$	semis	$\frac{1}{2}$ as	S	S
$\frac{1}{12}$	uncia	$\frac{1}{12}$ as	–	/
$\frac{1}{24}$	semuncia	$\frac{1}{2}$ uncia	≷	S·
$\frac{1}{48}$	sicilicus	$\frac{1}{4}$ uncia	ↄ	ↄ
$\frac{1}{72}$	sextula	$\frac{1}{6}$ uncia	∿	∪
$\frac{1}{144}$	dimidia sextula	$\frac{1}{2}$ sextula	≯	Ψ
$\frac{1}{288}$	scripulum	$\frac{1}{24}$ uncia	≫	
$\frac{1}{3}$	triens	4 unciae	= =	3S
$\frac{1}{4}$	quadrans	3 unciae	= –	3·
$\frac{1}{6}$	sextans	2 unciae	=	3
$\frac{1}{8}$	sescuncia	$1\frac{1}{2}$ unciae	≷–	S·
$\frac{1}{9}$	nona	1 uncia, 2 sextulae	–∿	/∪∪
$\frac{5}{12}$	quincunx	5 unciae	= = –	3S·
$\frac{7}{12}$	septunx	$\frac{1}{2} + \frac{1}{12}$	S –	S
$\frac{2}{3}$	bessis	$\frac{8}{12}$	S =	SS
$\frac{3}{4}$	dodrans	$\frac{9}{12}$	S = –	SS
$\frac{10}{12}$	dextans	$\frac{10}{12}$	S = =	SSS
$\frac{11}{12}$	deunx	1 as – 1 uncia	S = = –	SSS

Fig. 25.8 Table of named Roman fractions according to Tropfke

A possible calculation is:

$$\left(1\frac{1}{3}\right)^2 \div \left(1\frac{1}{4}\right)^2 = \left(\frac{4}{3}\right)^2 \times \left(\frac{4}{5}\right)^2 = \left(\frac{16}{15}\right)^2 = \left(1+\frac{1}{15}\right)^2 = \left(1+\frac{1}{5}\cdot\frac{1}{3}\right)^2$$

$$\Rightarrow = 1 + 2\cdot\frac{1}{5}\cdot\frac{1}{3} + \left(\frac{1}{5}\cdot\frac{1}{3}\right)^2 = 1 + \frac{2}{5}\,\text{triens} + \frac{1}{25}\,\text{triens}^2$$

The intermediate calculations required are:

$$\text{triens}^2 = 16\ \text{dimidiae sextulae} = 32\ \text{scrupuli}$$

$$\frac{2}{5}\,\text{triens} = 1\ \text{uncia} + \frac{1}{5}\,\text{quadrans} = 1\ \text{uncia} + \text{semuncia} + \frac{1}{10}\,\text{uncia} = \frac{1}{8} + \frac{24}{10}\,\text{scripuli}$$

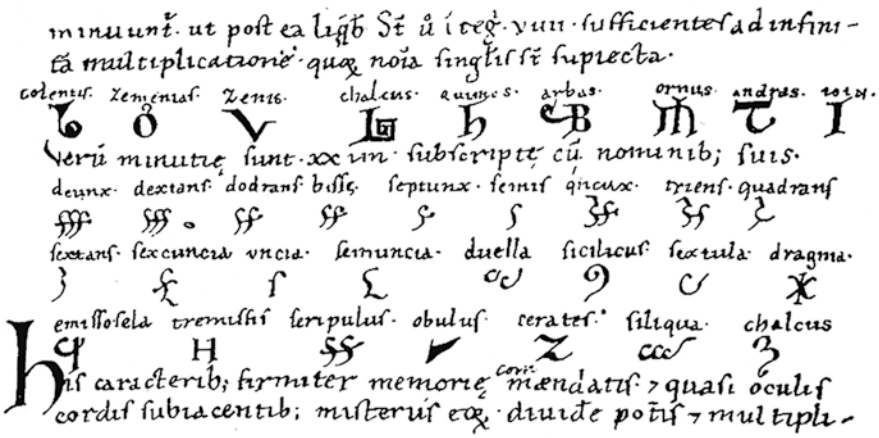


Fig. 25.9 Roman numerals from a Boethius manuscript (Wikimedia Commons)

In Roman numerals this gives

$$\left(1\frac{1}{3}\right)^2 \div \left(1\frac{1}{4}\right)^2 = 1 + \frac{32}{25}\ \text{scripuli} + \frac{1}{8} + \frac{24}{10}\ \text{scripuli} \approx 1 + \frac{1}{8} + 3\frac{2}{3}\ \text{scripuli}$$

$$\left(\frac{32}{25} + \frac{24}{10}\right)\ \text{scripuli} = 3\frac{34}{50}\ \text{scripuli} \approx 3\frac{34}{51}\ \text{scripuli} = 3\frac{2}{3}\ \text{scripuli}$$

The verbal result of the calculation is therefore: *unum et octava hoc est sescuneia et scripuli tres et bes scripuli,* in modern notation:

$$\left(1\frac{1}{3}\right)^2 \div \left(1\frac{1}{4}\right)^2 = 1 + \frac{1}{8} + \frac{3\frac{2}{3}}{288}$$

Some (later) Roman numerals for fractions can be found in the Boethius manuscript (BM Arundel 343) from the 12th century (Fig. 25.9).

25.2 Mathematical Examples from Literature

A well-known example comes from the poet *Horace* (Ep. 2.1.69–71), who in his old age still remembered very well how, as a boy, he was dictated the songs of Livius Andronicus by his teacher. His teacher was L. Orbilius Pupillus, who is remembered as "generous with the beatings":

> carmina Livi … memini quae plagosum mihi parvo Orbilium dictare.

In the following excerpt from a poem, Horace (*Ars poetica,* 323–330) makes fun of Roman education; he is probably thinking of his own school days:

> Grais ingenium, Grais dedit ore rotundo
> Musa loqui, praeter laudem nullius auaris;
> Romani pueri longis rationibus assem
> discunt in partis centum diducere. 'Dicat
> filius Albini: si de quincunce remota est
> uncia, quid superat? Poteras dixisse.' — 'Triens.' — 'Eu!
> Rem poteris seruare tuam. Redit uncia, quid fit?' 'Semis.'

> The Muse gave the Greeks talent and the ability to speak with a round mouth [=melodiously],
> they are greedy for nothing but glory;
> Roman boys learn to divide an as into a hundred parts by long calculations. 'You, son of Albinus, tell me:
> If one twelfth is taken away from five twelfths,
> what remains? – Speak!' – ‚A third.' – ‚Good!
> You will know how to preserve your property. One twelfth is added, what happens?' 'A half.'

The following calculation is being made here:

$$\frac{5}{12} - \frac{1}{12} = \frac{1}{3} \therefore \frac{5}{12} + \frac{1}{12} = \frac{1}{2}$$

Since the Romans did not have names for the hundredths mentioned by the poet, G. Friedlein (1866) tried to reconstruct a corresponding calculation that provides a decomposition into available fractions:

$$\frac{1}{100} = \frac{144}{14\,400} = \frac{100+44}{14\,400} = \frac{1}{144} + \frac{44}{14\,400}$$

$$\frac{44}{14\,400} = \frac{132}{43\,200} = \frac{1}{432} + \frac{32}{43\,200}$$

$$\frac{32}{43\,200} = \frac{128}{17\,200} = \frac{1}{1728} + \frac{28}{172\,800}$$

$$\Rightarrow \frac{1}{100} = \frac{1}{144} + \frac{1}{432} + \frac{1}{1728} = \frac{1}{72}\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{24}\right)$$

The last fraction with a denominator of 172 800 is neglected. Given this amount of calculation, one wonders whether Horace's hundredths are not after all an invention of the poet.

Gaius Plinius Secundus Maior, called Plinius the Elder, author of a 37-volume *Natural History (Naturalis Historia),* gives distances in steps *(passus)* for land surveying; however, since 1000 *(mille) passus* are exactly one mile, his statements can be converted directly into Roman miles. One example is the "width" of Africa:

$\overline{|XVIII|}\ \overline{LXXV} = 1\,875\,000$ (passus) $= 1875$ (miles)

In Book VI of his *Natural History*, Pliny (Hist. Nat. VI, 38) wants to determine the size of the continents:

> $\overline{|XXXVII}\ \overline{XLVIII}$. $\overline{|XXXVII|}\ \overline{XCVIII}$, $\overline{DCCL}\ \overline{DCCCCX}$ Agrippa made, $\overline{|XLVII|}\ \overline{VIII}$, $\overline{|LX|}\ \overline{III}$, $\overline{|XVIII|}\ \overline{LXXV}$ Est ergo ad hoc praescriptum Europae magnitudo … longitude. Africae (ut media ex omni varietate prodentium computatio) efficit longitudo latitudo, qua colitur, nusquam excedit. Sed quoniam in Cyrenaica eius parte eam fecit Agrippa, deserta eius ad Garamantas usque, qua noscebantur, completens, universa mensura quae veniet in comparationem efficit. Asiae longitudo in confesso est DCCL, latitudo sane computetur ab Aethiopico mari Alexandriam iuxta Nilum sitam, ut per Meroen et Syenen mensura currat.
>
> According to this regulation, the size of Europe [position corrupted], the length 8148 miles. From Africa—to take the mean of all the different estimates - the length 3798 miles, the width, as far as it is inhabited, does not exceed 750 [miles] anywhere; but since Agrippa has fixed the width at 910 miles by including his deserts up to the Garamantes, as far as they were known, the total amount that is accounted for here is 4708 miles. The length of Asia is, according to all concession, 6375 miles; but the width is calculated from the Ethiopian Sea to Alexandria, which is located on the Nile, so that the measuring line runs through Meroë and Syene, to 1875 [miles].

It is astonishing that Pliny *adds* (!) the width of Africa to its length here and obtains 4708 as the "area". Or is this the well-known error that often occurred in antiquity, that one could infer the area of a figure from its circumference, which was only uncovered by the speaker Marcus Fabius Quintilianus[14]? He formulated it as follows: The claim *figures with equal circumference have equal area* is false, because it actually applies that *the circle has more area than the square with equal circumference.* By the way, Quintilian was the first rhetoric teacher paid by the state (around 60 AD). He demanded more pedagogy in the training of speakers and pleaded for a mathematical education (Inst. Orat. I, 10, 34):

> Geometry should be learned at a young age, because from there comes the mobility of the mind, the sharpness of the intellect and the speed of perception (*cedunt agitari namque animos et acui ingenia et celeritatem percipiendi venire inde concedunt*).

Later he explains:

> Geometry is divided into two parts: on the one hand it is concerned with numbers, on the other hand with figures. Now the knowledge of the former is not only necessary for the speaker, but for everyone who has at least received elementary education. Such knowledge is often required in real cases where the speaker lacks education. I will not say anything if he hesitates when recalculating, but there must be no contradiction between what he says with words and what he expresses with insecure or inappropriate gestures [when calculating with his fingers].

[14] Quintilianus M. F.: Institutiones oratoricae, Ed. Halm, Band 1, Leipzig 1868, S. 62.

The fraction calculation provided by Pliny shows that fractions without names were represented as sums of basic fractions in the Roman system and that approximations were accepted:

> It becomes clear that Europe is slightly less than half the size of Asia, but twice as large and one sixth the size of Africa. If you add up all these values, it becomes clear that Europe makes up a third of the whole earth and slightly more than an eighth [part], Asia a fourth and fourteenth, Africa a fifth and sixtieth.

Pliny determines the shares of the three continents in the earth's surface as follows:

$$\text{Europe} \geq \frac{11}{24} \therefore \text{Asia} \geq \frac{9}{28} \therefore \text{Africa} \geq \frac{13}{60}$$

These parts sum up to one $\left(\geq \frac{279}{280}\right)$. If you calculate the "area" of Asia analogously to Africa, you get 6888 ¾. From the numbers for Asia or Africa, the "surface" of the earth results here:

$$\frac{6888.75}{\frac{9}{28}} = 21\,431.7 \therefore \frac{4708}{\frac{13}{60}} = 21\,729.2$$

The agreement of both values is not too good, but may be a result of the accuracy of measurement at that time.

Although the following problem is not mathematically stated correctly, it was adopted in almost all medieval textbooks, as in the *Practica Ratisbonensis* (No. 35):

> A father on his deathbed has a pregnant wife and plans his will: If the child is a boy, he will receive 2/3 of the inheritance, 1/3 the woman. If the child is a girl, she will inherit 1/3, the woman 2/3. But now a twin pair (boy and girl) is born. How is the inheritance to be divided?

The problem cannot be solved mathematically, but only legally. The lawyer *Salvianus* Julianus, who brought the rights of freedmen into legal form under Hadrian, had a seventh division, whereby the boy should receive four, the girl two and the woman one share. M. Cantor already pointed out that this decision does not at all correspond to the sense of the will.

Gaius *Petronius* (Plinius the Elder calls him Titus) was governor in Bithynia and consul. Although he belonged to Nero's circle, he was driven to suicide by intrigues. He wrote the first Roman novel *Satyricon libri,* which is not completely preserved. The core of the novel is the "banquet of Trimalchion" (Cena Trimalchionis; 58,7); at this feast, about two dozen parvenus and freedmen meet and chat. One of the guests is the freedman *Hermeros,* who mocks school and higher education while speaking in vulgar Latin:

> Non didici geometrias, critica et *alogas menias*, sed lapidarias litteras scio, partes centum dico ad aes, ad pondus, ad nummum. (I did not learn geometry, literary criticism and the

> nonsense at Homer; but I can read stone inscriptions and say the percentages by small change, weight and cash).

The vulgar Latin uses words like "alogos menia"; "ἄ-λογος" is the Greek word "nonsensical", "menia" is probably a distortion of the beginning of the Iliad: "Μῆνιν ἄειδε, ϑεά …" (Sing the wrath, o goddess …).

The poet Decimus Iunius Iuvenalis *(Juvenal)* was a rhetorician and only became an author in old age, since his friend Martial does not mention his writing. Juvenal became popular through his proverbs; some are well known as sayings:

> It is difficult not to write a satire (I, 30).
> It is to be wished that a healthy mind dwells in a healthy body (X, 356)(Latin: Mens sana in corpore sano).
> In Rome, everything can be had for money (III, 183).
> The people urgently want two things: bread and games (X, 81)(Latin: panem et circenses)

He wrote 16 satires in five books; in the tradition of Horace, he denounces the moral corruption and hypocrisy of society, in particular the upper class. Satire IX is called by Chr. Schmitz[15] "Lament of an aging gigolo". Here Juvenal mocks a certain *Naevolus,* who even has to ask for an abacus to calculate the fee for his love services (IX, 27):

> He calculates and flatters. An invoice is to be created, he calls the young slaves with the abacus: Calculate 5000 sesterces for everything, then my efforts should be taken into account.

When Marcus Tullius *Cicero* was still a young lawyer, he had the opportunity to profile himself as a lawyer in a lawsuit by the Sicilian provinces. It was about the abuse of office against the retired corrupt governor Gajus *Verres*. In his speech against Verres[16] (II.3.49) he calculated Verres' side income from the sale of grain and showed his skill in juggling with large numbers:

> Professio est agri Leontini ad iugerum $\overline{XXX}$; haec sunt ad tritici medimnum $\overline{XC}$, id est mod. $\overline{DXXXX}$; deductis tritici mod. $\overline{CCXVI}$, quanti decumae venierunt, reliqua sunt tritici $\overline{CCCXXIIII}$. Adde totius summae *DXXXX* milium mod. tres quinquagesimas; fit tritici mod. $\overline{XXXII}CCCC$ (ab omnibus enim ternae praeterea quinqugesimae exigebantur) sunt haec iam ad $\overline{CCCLX}$ mod. tritici.
>
> The circumference of the arable land of Leontinoi is about 30 000 acres; this amounts to about 90,000 Greek or 540,000 Roman bushels of grain. If we subtract from this the 216 000 bushels that were leased as a tax quota, there remains a balance of 324 000 bushels. But we must now add 6 percent to the total amount, that is, to the 540 000 bushels (for this 6 percent was taken from everyone without exception, in addition to the rest); this already amounts to 360 000 bushels of grain.

[15] Schmitz Chr.: Juvenal, S. 122, Georg Olms Verlag 2019.

[16] Cicero M. T.: Orationes in Verrem, Friedrich Spiro (ed.), e-artnow, 2014.

Fig. 25.10 Roman school scene (relief tomb Trier/Wikimedia Commons)

Sect. 25.3 on the Roman school shows that mathematics did not have a special status.

25.3 The Roman school

Schools existed according to Livius (III, 33, 6) already a long time, probably since Etruscan times. According to Plutarch (Quaest. Rom. 59, 278E) there is the school since the 3rd century BC. The Latin word for school (*schola*) derives from the Greek word σχολέ (= leisure time). At the age of about 7 to 11 or 12 years old boys and girls visited the elementary school *(ludus litterarius),* which was privately organized. There was no school attendance; the teacher *(ludi magister)* had to collect the school fees himself, as an edict of Marcus Aurelius requires. The lessons took place mostly in a private shop or in a public hall. Fig. 25.10 shows an idealized school scene: The bearded teacher (in Greek style) lectures—sitting on an elevated seat—between two students, who in turn hold open scrolls in their hands. On the right appears a latecomer student with raised hand in greeting.

The school year began in March and was interrupted in the summer months; there was no free weekend, but many holidays, especially in the late period. Writing, reading and elementary arithmetic were learned, rules were memorized, Augustine writes: *Ubi legere et scribere et numerari discitur*. It probably didn't go without corporal punishment; the image from Pompeii is well known, in which the teacher swings the rod *(ferula)*. The physical punishment was greatly hated by the students, as *Augustinus*[17] (Confess. I, 9) reports:

> For even when I was still a boy, … I prayed to you [God], still small, but with great earnestness, that I would not get beaten in school, and since you did not answer me, which

[17] Augustinus A., Lachmann O. (Hrsg.): Die Bekenntnisse, e-artnow 2015.

> was for my salvation, the adults mocked, even my parents, who only wanted my best, the beatings I got.

Not all students liked memorization, Augustinus (Confess. I, 13) writes:

> But now '*one and one is two, two and two is four*' and so on was a song of very hateful sound to me and the most pleasant sight for my vanity was the wooden horse of the armed men, the burning of Troy and the shadow of Creusa [wife of Aeneas].

The next higher level was the grammar school. The curriculum was based on the Greek-Hellenistic education; the teacher was called *grammaticus* or *litterator,* his office required a higher qualification. Homer, the Greek tragedians and Latin authors such as Terence and Vergil were read and interpreted. Even for willing students it was not easy to learn Greek; so Augustine confesses his unwillingness to learn Greek:

> But how it actually came about that I hated Greek literature, I myself am not quite clear. For I loved Latin literature, not as the elementary teachers, but as the so-called grammarians taught it; for that elementary instruction was no less burdensome and embarrassing to me than all things Greek.

The students attended the grammar school until they turned 17, the age at which they donned the *toga virilis*. In Diocletian's Price Edict[18] the maximum school fee that a teacher could charge per student was set:

- Elementary school teacher 50 Denare,
- Arithmetic teacher 75 Denare,
- Upper school or geometry teacher 200 Denare,
- Rhetor 275 Denare.

It was not until Christianity became the state religion that the state took over the operation of the grammar schools. After an edict by Theodosius II, private lessons were banned (Cod. Theod. 14, 9, 3).

The third and highest level was the school of the *Rhetor* or *Orator.* The main subject of instruction was rhetoric, with basic training also provided in philosophy and jurisprudence, so that the students would be armed with this knowledge for their future careers as lawyers, officers, consuls, or speechwriters. In addition to theoretical lectures, famous speeches by Demosthenes and Cicero were discussed, and practical declamation exercises and conversation with the teachers were practiced. There existed schools that were maintained by municipalities or imperial sponsors and therefore usually better equipped. Parents with better financial resources could send their sons to Athens, Rhodes, or Alexandria for such external training—considered the crowning achievement of an academic education.

[18] www.hs-augsburg.de/~harsch/Chronologia/Lspost04/Diocletianus/dio_ep_i.html [12.09.2019].

Conversely, many scholars also came to Rome to teach. Famous philosophers included *Plotinus* (in Rome from 245) and later *Porphyry* (in Rome from 263). The latter took over the (neo-Platonic) philosophy school from Plotinus. For conservative circles, these immigrants were unwelcome; for example, the famous censor Cato the Elder had already deported a whole group of immigrant philosophers in 155 BC.

25.4 The Role of Roman Surveyors

Approximately since the middle of the 1st century AD there were professional surveyors (*Agrimensores* or *Gromatici*). The name *gromaticus* derives from the used measuring instrument *groma*. The writings of the Agrimensoren were often copied and later in the 6th century AD collected to a *Corpus agrimensorum*. Authors of early surveyor writings are mentioned:

- Marcus Terentius Varro (116–27 BC),
- Lucius Iunius Columella (about 65 AD),
- Sextus Iulius Frontinus (ca. 40–103 AD),
- Balbus (about 107 AD).

Later authors from the 2nd century AD are:

- Hyginus (Gromaticus),
- Siculus Flaccus,
- Marcus Iunius Nipsus,
- Epaphroditus,
- Vitruvius Rufus.

A survey of the authors is provided by the article by Pepa Castillo[19].

The manuscript *Hygini Gromatici Liber* was read as the book of *Hyginus* Gromaticus; meanwhile it is known that *Liber Gromaticus* refers to the book. However, the nickname Gromaticus was often retained in order to distinguish Hyginus from his namesake, the fabulist Hyginus *Mythographus*. Although the transmitted manuscripts in the Middle Ages are often beautifully designed, the texts still contain some spelling mistakes and inaccuracies. This is a consequence of the ignorance of the early medieval copyists: partly the Latin was only imperfectly understood, some technical details and terms were not understood at all. As an example of such a late antique Latin text (with corrections in square brackets) the following quotation from the surveyor Agennius *Urbicus* (De controversiis agrorum, 22.7–8) is given:

[19] Castillo P.: The *conversia de iure territorii* among the Gromatikern, pp. 149–170, in the anthology Knobloch-Möller.

> Omnium igitur honestarum artium, quae sive naturaliter aguntur sive a[d] naturae imitationem proferentur, materiam optinet rationis artificium geometria, principio ardua ac difficilis incessu, delectabilis ordine, plena praestantiae, effectu insuperabilis. manifestis enim rationi[bu]s executionibus declarat [rat]ionalium materiam, ita ut geometria[m] inesse artibus aut arte[s] ex geometria esse intelligat[ur].
>
> Of all the liberal arts, which are carried out according to nature or in their imitation, geometry receives its object from the artist's plan; it is at first laborious and difficult to access, pleasant in its regularity, full of beauty, unsurpassed in its effect. In its clear guidelines, it illuminates rational thinking, so that one understands that geometry belongs to the arts and the arts can be understood from geometry.

In his intended renovations of the Roman Empire, Caesar planned both a land survey and a calendar reform. His violent death prevented the survey project; it was later carried out by his nephew Octavian, the future emperor Augustus. The latter commissioned the general Vipsanius *Agrippa* and the land surveyor *Balbus,* who had proved themselves militarily, with this task. A secondary purpose was the creation of an oversized, true-to-scale map of the *Orbis Romanus,* which was to be displayed to the public. For military purposes, it was particularly important to determine the distance between two Roman settlements; from this, the general could determine the number of days' marches required.

Such a road map represents the *Tabula Peuteringeriana*, a 12th century copy of a late Roman map (after 330 AD). It probably comes from a monastery, was rediscovered by Conrad Celtis, who handed over the table to the Augsburg Conrad Peutinger (1493–1547) with the order of preparing the table for printing. Fig. 25.11 shows a section at the top of the map showing the connection of the cities of Bonn *(Bonnae),* Cologne *(Agrippina)* and Xanten *(Colonia*

Fig. 25.11 Tabula Peuteringeriana (detail/Wikimedia Commons)

Traiana)(Provincia Germania). The number of "corners" indicates the number of days required for the journey.

At Caesar's time there was a difference of 67 days between the official calendar and the astronomical calendar, as Cassius Dio (43, 26) reports. Sosigenes' proposal to immediately change the calendar to the correct astronomical date was rejected by Caesar. The changes took place over long years, so that only little was changed in the old Roman calendar.

It was the task of the surveyors to survey and legally divide conquered land, as *Siculus* Flaccus writes in his work *De conditionibus agrorum*:

> The surveyed land is taken from the defeated enemies and divided up among the victorious soldiers and veterans (*ex hoste uictori militi ueteranoque assignatus hostibus pulsis*).

He knew of the disputes that took place between the occupiers and the defeated during the land grab, and therefore wrongly derived the word *territorium* etymologically from *terror*.

In early times, the land grab was subject to a religious rite, as the legend of the founding of the city of Rome by Remus and Romulus tells. Augurs were commissioned to explore the will of the gods by observing the flight of birds. If the gods were inclined, the outer boundary of the future settlement was fixed in a ceremonial procession by means of a ploughed furrow, with the plough being pulled by two oxen. If a wall is built over this furrow, it is sacred. When Remus carelessly jumped over the wall erected by Romulus, he violated a divine command and was killed by Romulus.

The properties were tax-free for new citizens or leased to rich private individuals. The division happened mostly in rectangular or trapezoidal form, which was oriented to given localities such as the existing road network, river courses, etc. The borders were mostly marked by special stone settings. For the displacement of a boundary stone, a fine of 5000 sesterces was imposed *(terminos singulos, quos eiecerit locoue mouerit)*. The mapping was documented by engravings in stone or incisions in copper plates.

Three parts of a cadastre—*A*, *B*, *C*—have been preserved from the city of Arausio (today Orange/France), which is carved into a marble slab. Fig. 25.12 shows fragment 7 of cadastre A. In the middle is a river with an island. It is striking that the roads run parallel to the Eygues river and not following the grid; they were probably created before the mapping. The original size of cadastre B could be reconstructed; it was 5,5 m × 7 m (after Dilke[20]).

In the area of *Villanova di Camposampiero* (Padua) in the Po Valley, the subdivision into the unit of area *centuria* as a square network of paths is still preserved today. The division can be seen in impressive completeness on *Google Earth*. The accuracy can be seen from the following message: Prof. H. Stohler (Basel) discovered an 80 km long, straight section of the Upper Germanic-Rhaetian Limes,

[20] Dilke O.A.W.: The Roman Land Surveyors, Newton Abbott, David and Charles 1971, p. 166.

Fig. 25.12 Roman cadastre of the city of Orange (fragment), (Wikimedia Commons)

which, according to a survey by Prof. E. Hammer (Stuttgart), is only subject to an average directional error of ± 1.9 m at a distance of 30 km!

The unit *centuria* mentioned above with a side length of 20 actus (plural actus) covers 50.4 ha, with 1 actus = 35.484 m.

The beginning of the work *Ad Celsum expositio et ratio omnium formarum* by *Balbus,* who also served under Trajan, is preserved. In it, he reports to his addressee *Celsus,* a professional colleague, in extravagant words, how he was of service to an imperial military expedition of the emperor:

> As soon as we had conquered the enemy's land, the [planned] construction of our Caesar's fortifications required the immediate start of the surveying work. In a given distance, two parallel lines had to be drawn, along which a huge palisade wall was to be erected for the protection of our operations. With the help of your invention [the dioptra], we were able to extend the parallels everywhere by means of sights. To plan the construction of bridges, we were able to determine the width of rivers on this side of the river, even if the enemy [on the other side] tried to interfere. Then this wonderful measuring method [the triangulation] also showed us how to determine the height of the conquered mountains. After long experience and participation in such undertakings, I began to appreciate this method of calculation more and more, like a ritual in the temple.

Cassiodorus (490–583), an official at Theodoric's court in Ravenna, still had knowledge of land surveying based on Heron's writings and wrote in sarcastic, official style (quoted according to Dilke, p. 45–46):

> Other sciences are so theoretical that the professors only have a handful of students. But the *agrimensor* is entrusted with the settlement of boundary disputes, which have arisen in order to put an end to frivolous quarrels. He is a judge of his own kind; his court is the deserted fields; one might think him crazy, seeing him running along winding paths. When he looks for evidence [like boundary stones] in the wild undergrowth or in the thicket, he does not walk like you and me, he chooses his own way. He explains his decisions, he applies his knowledge, he decides disputes by his footprints, and like a giant stream he takes away a piece of land from one and gives it to the other.

In the following letter, Pliny d. J. (then governor of Bithynia) suggests to Emperor Trajan that land surveyors should also be used in the provinces to control finances.

> I am currently examining the finances of the city of *Prusa,* expenditures, income and assets, the deeper I look into the books, the more convinced I am of the need for inspection. I am writing this report, Emperor, immediately after my arrival here.
>
> I entered my province on September 17 and found general obedience and loyalty, a tribute of the community to you. Consider, Emperor, sending a land surveyor? After reliable inquiries have been made, it could, in my opinion, collect considerable sums of money here from the officials who hold public office. I am convinced of this since I have been holding the books of Prusa.

Digression Pliny the Younger was the nephew of Pliny the Elder, who adopted him after the death of his sister. We know more details about his life than that of any other Roman person—with the exception of Cicero. All of his letters (369 in total, compiled in ten volumes) are written in such a way that every passage is quotable and each letter sounds like an official report. The letters (VI, 16) (VI, 20) to *Tacitus,* in which he reports the death of his uncle during the eruption of Vesuvius, as well as the letter (X, 96) to the emperor, in which he consults him on the procedure for the persecution of Christians, are particularly famous. Tacitus was in close contact with the older Pliny and used his notes on the campaigns against Germania when writing his work *Germania* ..

25.5 From the Corpus Agrimensorum

The mathematics of the Roman land surveyors is, in the opinion of M. Folkerts[21], languishing in the shadow of mathematics history. He cites two reasons for this. On the one hand, it was the strong influence of M. Cantor[22], who wrote a seminal work on the land surveyors. Since Nicolai Bubnov[23] contradicted his view of the geometry of Pseudo-Boethius, Cantor tried to suppress his work. On the other hand, Bubnov's work is difficult to read because it contains many Latin quotations. Furthermore, the numerous corrections can only be found in the appendix; so when reading an article , one must check in the extensive appendix whether it was corrected later. A more recent article by Folkerts[24] can be found in the anthology of Knobloch/Möller.

[21] Folkerts M.: Mathematical problems in the Corpus agrimensorum 1992, pp. 25–27, in the collected works of Folkerts.

[22] Cantor M.: The Roman land surveyors and their position in the history of surveying, Leipzig 1876.

[23] Bubnov, N. (ed.): Gerberti, postea Silvestri II papae, Opera Mathematica 972–1003, Berlin 1899.

[24] Folkerts M.: The mathematics of the land surveyors 2013, pp. 131–148, in the collected works of Knobloch/Möller.

The *Corpus Agrimensorum* is a collection of various writings of the Roman land surveyors, which was compiled in the 6th century. The work is available in two main manuscripts representing different editions: Codex *Arcerianus* (A), (B) (Wolfenbüttel, 6th century) or Codex *Palatinus* (P) (9th century). Arcerianus A was discovered in 1495 in the monastery *Bobbio* (Province of Piacenza), where *Gerbert* von Aurillac was the abbot and used its library intensively. The name of the Codex derives from Johannes Arcerianus, who acquired the Codex in 1566. Important copies and transcriptions are the following:

- Manuscripts containing mixed texts from both main manuscripts can be found in Florence (F, end of 9th century) or in Erfurt (E, 11th century), with E being more extensive than F.
- Transcriptions of (P) are available in Wolfenbüttel (G) or Brussels, the latter not containing any pictures.
- Transcriptions of (A), (B) exist in the Vatican (V) or in Jena (J), both of which differ in arrangement and text fidelity.

Arcerianus A contains 33 individual texts; individual parts of it appeared in 1491 in print, the complete codex only in 1607. The print contains the following texts:

1. two fragments that may originate from Varro (116–27 BC),
2. Balbus' treatise *Expositio et ratio omnium formarum* from the time of Trajan,
3. the so-called *Liber podismi* (anonymous),
4. treatise of the (otherwise unknown) authors *Epaphroditus and Vitruvius Rufus,*
5. treatise *Fluminis Varatio* (determination of river width),
6. treatise *De iugeribus metiendis* (land surveying),
7. excerpts from the first books of Euclid's *Elements,*
8. treatise *De re rustica* (On agriculture) by Columella.

The writings 1) to 6) can be found in Bubnov, writing 8) is in two parts in the work[25]. Important editions come from F. Blume[26] and others, H. Butzmann[27] (1970), C.O. Thulin[28] (1913). A new, comprehensive edition is available in English from B. Campell[29].

For all Latinists a task in the original showing the use of the Roman numbers:

[25] Columella, Löffler K. (ed.): De re rustica, volume I + II, Litterarischer Verein, Stuttgart 1914.

[26] Blume F., Karl Lachmann K., Rudorff A. F. (ed.): Gromatici veteres. The writings of the Roman land surveyors, 2 volumes., Berlin 1848–1852.

[27] Butzmann H. (ed.): Corpus agrimensorum Romanorum, Leiden 1970.

[28] Thulin (ed.): Corpus agrimensorum Romanorum, Leipzig 1913.

[29] Campbell B. (ed.): The writings of the Roman land surveyors, London 2000.

Si fuerit arca longa ped. XXX, lata ped. XV, alta ped. VII, duco longitudinem: fiunt ped. CCX. Hoc duco per latitudinem: fiunt ped $\overline{III}CL$.

If a cistern is 30 *feet* long, 15 *feet* wide and 7 *feet* high, I multiply the length with the height: this makes 210 [square] *feet*. With the width multiplied, this results in 3150 [cubic] feet.

The determination of the content of a wine or lime barrel is described as follows: First, the three diameters of the barrel are measured at the bottom, in the middle and at the top. From this, the sum is formed, 1/3 of the sum is squared and multiplied by 11/14 of the barrel height. Here one suspects that the formation of an average was not understood.

The traditional figured numbers were misunderstood and thus interpreted as surface formulas. M. Folkerts explains the origin of the strange surface formulas by the misapplication of the recursion formula for figured numbers. The formula for the k-th polygonal number of the n-sided figure is:[30]

$$P_n(k) = \frac{1}{2}\left[(n-2)k^2 - (n-4)k\right]; n \geq 3, k \geq 1$$

If one erroneously sets k to the side a of the regular n-sided figure, one obtains

$$P_n(a) = \frac{1}{2}\left[(n-2)a^2 - (n-4)a\right]; n \geq 3, a \geq 1$$

For the triangle, pentagon and hexagon, the nonsensical formulas result

$$P_3(a) = \frac{1}{2}\left(a^2 + a\right); P_5(a) = \frac{1}{2}\left(3a^2 - a\right); P_6(a) = \frac{1}{2}\left(4a^2 - 2a\right)$$

Digression

The use of figured numbers in determing areas (apart from the wrong dimension) also caused computational problems. Thus, for the surface determination of the equilateral triangle (side $a = 7; h = \frac{6}{7}a = 6$), the triangle number $P_3(a) = \frac{1}{2}a(a+1)$ was used. The result $F_3(7) = 28$ contradicts the surface calculation as half a rectangle $F = \frac{1}{2}ah$, which results in 21.

This was the problem that was brought to him by his student Adelbold, Bishop of Utrecht, but with different numbers: $a = 30; h = \frac{6}{7}a \approx 26$. Which result then gives the true area?

Gerbert[31] had the following *notable* idea:

He layed the triangle on a grid of unit squares (Fig. 25.13) and obtained (with a square at the top) 28 squares. This explains the result of the triangle number $P_3(7)$. For the usual area measurement, all outer parts of the grid squares would be cut off; here therefore twelve (on average) half squares plus the one at the top.

[30] "Boethius" Geometry II, Folkert M. (ed.): Steiner Verlag 1970, p. 101.

[31] Sigismondi C.: Gerbert of Aurillac: Astronomy and Geometry in Tenth Century Europe, arXiv: 1201.6094v1.

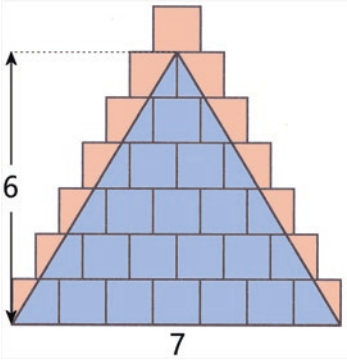


Abb. 25.13 Settlement of Anxur-Tarracona from the manuscript of Hyginus (Wikimedia Commons)

Seven squares are thus omitted; this explains the (usual) result $F = 21$. The letter to Adelbold ends:

> To understand this better, you have to look at this figure and *always keep me in mind*.

Fig. 25.14 from *De limitibus constitionibus* by Hyginus Gromaticus shows the square grid for the settlement of Axurnas in the Anxur-Tarracona region, founded in 329 BC (according to Dilke, p. 218). The mapping is parallel to the *Via Appia.* Earlier consuls like *Appius* Claudius Caecus (312 BC) could become famous if the roads or aqueducts whose construction they promoted were named after them.

Fig. 25.14 Interpretation by Gerbert

A) From the fragments of Varro

Marcus Terentius Varro, born in 116 BC in Rome, was quaestor in 86, came to Athens for study in 82, was tribune of the people in 70, and praetor in 68 BC. For a long time Varro was a follower of Pompeius; after the latter's death he was pardoned by Caesar and charged with the construction of a state library. After Caesar's death he was outlawed, he escaped the death penalty and died in 27 BC. Cicero [Academia I] calls Varro *a man who is connected to us by shared studies and old friendship.*

His writings are not preserved, but were still known in the time when the Corpus Agrimensorum was compiled. So Cassiodorus could still provide some quotes from Varro's writings. Some fragments from the Codex Arcerianus are attributed to Varro; if this attribution is correct, these are the *oldest* pieces of Roman mathematics! They were collected by Bubnov in his book on Gerbert in Appendix VII. It is noteworthy that Varro—like Diophantus—used Greek words for powers of the unknown: *dynamus* (square), *kybus* (cube) and *dynamodynamus* (4th power).

First, Varro explains the common length and area measures:

- Finger width *(digitus)* 1.85 cm,
- Hand width (*palma* = 4 finger widths) 7.4 cm,
- Foot (*pes* = 16 finger widths) 29.57 cm,
- Cubit (*cubitus* = 3/2 foot) 44.63 cm,
- Step (*gradus* = 2 1/2 foot) 73.93 cm,
- Double step *(passus)* 1.479 m,
- Mile *(mille passuum)* 1.479 km.

Area measures are:

- Actus = 144 cubits^2 = 14 400 feet^2,
- iugerum = 2 actus = 28 800 feet^2 = approx. 1 acre.

Here are some tasks from Varro:

1. I take this (14 feet) three times, it is 42. I add 2 [= 1/7 of the diameter], it is 44: this is the circumference.

   > Determine the circumference from the diameter. Do it like this: Take the diameter 14 feet three times, it is 42. Add 2 [= 1/7 of the diameter], it is 44: this is the circumference.

The Archimedean value is used here $\pi \approx 3\frac{1}{7}$.

2. The same in another way: Multiply the diameter by 22, it is 308; take 1/7, it is 44.
3. The diameter is calculated from the circumference by dividing by 22 and multiplying by 7.

4. A right-angled trapezoid *(ager cuneatus)* is 100 feet long, 130 feet on one side, 70 feet wide on the other. The area is sought in *jugera.* The calculation is $\frac{70+130}{2} \cdot 100$ (square) feet. Conversion to jugera: It is successively divided by 100, 24 and 12. Result $\frac{1}{2}$.
5. Area of an isosceles triangle of side 40 [feet]. The calculation for the height 34 is unclear, it was apparently rounded. Area is $40 \cdot \frac{34}{2} = 695$. Using the approximation $\sqrt{3} \approx \frac{26}{15}$ gives the height $34\frac{2}{3}$.
6. The triangle (13, 14, 15). Here the height 12 is correctly determined, but without calculation. The triangle is heronian.
7. Polygonal numbers:
   Triangle: Take the side, square it, add the side itself and take the half. Here the polygonal number $\frac{1}{2}\left(a^2 + a\right)$ is used as an area measure.
   Pentagon: Take the side, square it, the result times 3, subtract the side itself and take the half.
   Decagon: Take the side, square it, the result times 8, subtract the side itself six times and take the half.
   Here is a task with a polygonal number:
8. A field has the length of 1600 feet. I multiplied its area [in jugera9 by the triangle and got the width.
   Calculation: 28800/1600 = 18, times 2 gives 36, one unit away 35. The corresponding triangle number is $\frac{1}{2} \cdot 35 \cdot 36 = 630$.
9. A field has a length of 1800 feet. I multiplied its area [in jugera] by the cube and got the width.
   It is calculated: $28\,800 \div 1800 = 16$; $\sqrt{16} = 4$; $4^3 = 64$. The width is 64 feet.
10. Ten-sided: Take the side, square it, take half, subtract the side [i.e. the root] from the result, add half the side. This gives the height. Then take the side four times and multiply by the height; that is the area.

B) Area calculations according to Columella

L. Junius Moderatus *Columella* came from Gades (today Cadix) and was a military tribune of the 6th armored legion, which was deployed for a long time in Syria. Returning to Italy, he devoted himself to agriculture. He wrote the book *De re rustica,* which was intended as a help for future land surveyors. Here are some of his calculations:

1. The area of a square with a side of 100 [feet] is $100^2 = 10\,000$.
2. The area of a rectangle with sides 240 and 120 is 28 800. He uses the formula that was already used by Egyptian surveyors for any quadrilaterals

$$A = \frac{a+c}{2}\frac{b+d}{2}$$

   The formula was also used for general triangles with the limiting case $d \to 0$.

3. A trapezoid with length 100 and widths 20 and 10 has an area

$$\frac{10+20}{2} \cdot 100 = 1500$$

   As a length, Columella measures the distance between the midpoints of the parallel sides. Therefore, the formula is only valid for isosceles trapezoids.
4. An equilateral triangle with side 300 has an area (here $\sqrt{3} \approx 26/15$)

$$\frac{1}{3} \cdot 300^2 + \frac{1}{10} \cdot 300^2 = 39\,000$$

5. A right triangle with sides 50 and 100 has an area $\frac{50 \cdot 100}{2} = 2500$.
6. A circle with a diameter of 70 has an area $\frac{11}{14} \cdot 70^2 = 3850$.
7. A semi-circle with a base line (*basis)* of 140 and a width *(curvaturae latitudo)* of 70 has an area

$$\frac{11}{14} \cdot 70 \cdot 140 = 7700$$

8. A circle segment that is smaller than a semi-circle has a base line of 14, and a width of 4. The area

$$\frac{1}{2}(16+4)4 + \frac{1}{14}\left(\frac{16}{2}\right)^2 \text{ little more than } 44$$

9. A [regular] hexagon with sides of 30 has an area of $\left(\frac{1}{3} \cdot 30^2 + \frac{1}{10} \cdot 30^2\right) \cdot 6$, which equals 2340.

To (4) The approximation $\frac{1}{3}a^2 + \frac{1}{10}a^2 = \frac{13}{30}a^2$ is probably from Heron. He writes, after all, about the equilateral triangle: *The height is equal to the side, minus 1/7.* This is based on the well-known root approximation

$$\sqrt{3} \approx \frac{12}{7} \Rightarrow h = \frac{\sqrt{3}}{2}a = \frac{6}{7}a$$

The approximation also transfers to the regular hexagon (9). As M. Folkerts has observed, the formula $a \cdot \frac{a}{2}$ is also used; it contains the crude approximation $\sqrt{3} \approx 2$. It can also be derived from the field measure formula for quadrilaterals, by setting $d \to 0$.

To (8): The approximation comes from Heron. If s is the chord length and h is the segment height, then the following applies.

$$A = \frac{1}{2}(s+h)h + \frac{1}{14}\left(\frac{s}{2}\right)^2$$

Since Heron could not derive the correct area formula, he chose the expression so that it gives an exact value in the case of the semi-circle. In this case, $s = 2r$ and $h = r$. The comparison with the exact formula shows

$$A = \frac{1}{2}(2r + r)r + \frac{1}{14}\left(\frac{2r}{2}\right)^2 = \left(\frac{3}{2} + \frac{1}{14}\right)r^2 = \frac{11}{7}r^2 = \frac{1}{2}\pi r^2 \Rightarrow \pi = 3\frac{1}{7}$$

Here is an additional interest calculation by Columella:

> The total price [of the loan] is 29 000 sesterces. In addition, there are interest rates of 6% per year, which amount to 3,480 sesterces in 2 years. This results in a total cost of loan and interest of 32 480 sesterces. The lender can claim 1 950 sesterces per year as an annuity on the total costs for the aforementioned 6%.

The first part of the calculation is exact: $29\,000\left(1 + 2 \cdot \frac{6}{100}\right) = 32\,480$. However, for the annuity $32\,480 \cdot \frac{6}{100} = 1948.80$. Here, a rounding probably took place, as $\frac{6}{100}$ was not correctly determined as the sum of Roman fractions. Columella's calculation is not apparent.

C) Further calculations from the Codex

a) Circular area: $A = d \cdot r \cdot \frac{11}{14}$ (d = diameter, r = radius)

b) Rhomboid area: $A = d \cdot h$; $h = \sqrt{a^2 - \left(\frac{d}{2}\right)^2}$ (a = side; d = diagonal)

c) Altitude of octagon: $h = \sqrt{\frac{s^2}{2}} + \frac{s}{2}$ (s = side)

d) Circle area: $A = \frac{U}{2} \cdot \frac{d}{2}$ or $A = \left(\frac{U}{4}\right)^2$ (U = circumference)

Diameter of the incircle of a right-angled triangle: $d = a + b - c$

e) Right-angled trapezoid: $A = \frac{a+b}{2} \cdot c$; $d = \sqrt{c^2 + b^2 + a^2 - 2ab}$ (c perpendicular to a, b)

f) Acute-angled triangle (Fig. 25.15): decomposition of the base by the foot of the altitude *(praecisura x)* according to Euclid (II, 13):

$$x = \frac{b^2 + c^2 - a^2}{2c}$$

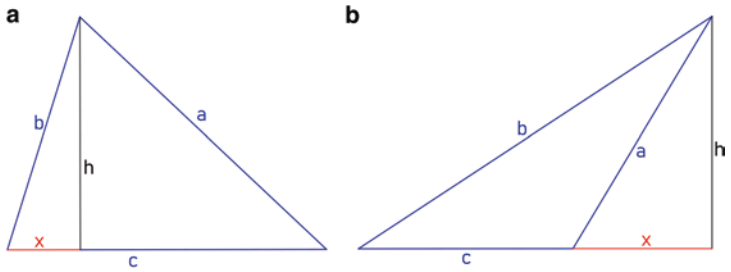


Fig. 25.15 Altitude in acute- and obtuse-angled triangle

g) Obtuse-angled triangle: extension of the base to the altitude point *(eiectura x)* according to Euclid (II, 12):

$$x = \frac{b^2 - a^2 - c^2}{2c}$$

The two last-mentioned formulas were used in particular to determine the altitude of the triangle by means of Pythagoras.

h) Spherical surface: $O = \frac{11}{14}(2d)^2$

The last three tasks can be found in a work by Epaphroditus. He also tried to calculate the "area" of a mountain: He measured the circumference (2500) at the foot of the mountain, the circumference (1600) in the middle of the mountain, and the circumference (100) at the top. The height is 500. The area results from the product of the average (of the circumferences) and the height.

D) Tasks of Nipsus

Marcus Junius Nipsus was probably not of Roman descent, as can be seen from the large number of ways his name was written (e.g. Nypsios). He may have been a freed Greek who served the family of the *Junier*. His lifetime is indefinite, most authors place him in the 4th century AD. If he was associated with the House of Junier, he lived in the 2nd century.

1. Given is a obtuse-angled triangle with the base 9 and the sides 17 and 10. From the top, the perpendicular is dropped upon the base; the extension *(eiectura)* of the base to the foot point is sought.

The following task M. Cantor calls somewhat euphorically *one of the most valuable in all geometry:*

2. Given a right-angled triangle with hypotenuse *c,* the sum of the catheti *a, b* and the area *A:*

$$c = 17 \therefore a + b = 23 \therefore A = 60$$

The manuscript calculates verbosely, what is expressed here with modern formulas:

$$(a - b)^2 = a^2 + b^2 - 2ab = c^2 - 4A \Rightarrow a - b = \sqrt{c^2 - 4A}$$

Adding the given sum $(a + b)$ yields the cathetus

$$a = \frac{1}{2}\left(\sqrt{c^2 - 4A} + a + b\right) = 15 \Rightarrow b = 23 - a = 8$$

This results in the triangle (15; 8; 17).

The specification 2) is overdeterminate; Nipsus also knows a terminate version here:

3. Given a right-angled triangle with hypotenuse c and the area A:

$$c = 25 \therefore A = 150$$

The calculation uses the following formulas (see above):

$$a \pm b = \sqrt{c^2 \pm 4A}$$

Thus:

$$a + b = \sqrt{625 + 600} = 35 \therefore a - b = \sqrt{625 - 600} = 5$$

This yields the sides

$$a = \frac{a+b}{2} + \frac{a-b}{2} = 20 \therefore b = \frac{a+b}{2} - \frac{a-b}{2} = 15$$

The desired triangle (20; 15; 25) is similar to the standard triangle (4; 3; 5). Also here the calculation runs purely verbal.

4. The acute triangle with sides (13; 14; 15) is considered, where the base 14 is. The division of the base by the foot of the altitude is sought. The same task can be found in the *Metrica* of Heron.
5. Nipsus can calculate Pythagorean triples such as (7; 24; 25), where the hypotenuse exceeds the larger leg by one. He writes:

> Datum numerum, id est III, in se. fit IX. hinc semper tollo assem. fit VIII. huius tollo semper partem dimidiam. fit IV. erit basis. ad basem adicio assem. erit hypotenusa, pedum V. (*Given the number 3, multiplied by itself it gives 9, minus 1 gives 8, halved gives 4, this is the [other] leg. 1 added gives the hypotenuse 5).*

He uses the formulas given by Heron (*Geometrica* 8, 1) here:

$$a^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2-1}{2} + 1\right)^2; a = 1 \bmod 2$$

$$a^2 + \left(\frac{a^2}{4} - 1\right)^2 = \left(\frac{a^2}{4} + 1\right)^2; a = 0 \bmod 2$$

With this, Nipsus could determine an integer triangle for every natural number $a \geq 3$ (as leg).

6. Nipsus also knows the Heron formula for the triangle area:

$$A = \sqrt{s(s-a)(s-b)(s-c)}; s = \frac{1}{2}(a+b+c)$$

He describes the calculation in words:

> To measure the area of any triangle—whether right-angled, acute-angled or obtuse-angled—using *one* method. It is found as follows: I take the three sides of any of the 3 triangles together. This is the case with the right-angled triangle, which is given by the cathetus 6 feet, the base 8 feet, the hypotenuse 10 feet. Combined, they give the number 24, from which I take half, makes 12. I set this and subtract the other numbers individually.

He writes the differences to 12, namely {6, 4, 2} in the lower row and multiplies the numbers below by 12.

> This makes 576. Taken from the root, it gives 24, this is the area. The content of all other triangles is obtained in the same way.

7. The decomposition of the hypotenuse of a right-angled triangle by the point of intersection of the altitude is sought. Nipsus also determines the altitude, which results as the product of the catheti, divided by the hypotenuse. Then the handwriting breaks off with the words:

Here ends the book of M. Junius Nipsus.

E) From the work of Frontinus

Sextus Julius Frontinus was—if one interprets Julius as a family name—a member of the noble Julier family; he lived approximately from 30 to 103 n. Chr. and served under the emperors Vespasian, Titus, Domitian, Nerva and Trajan. In 70 Frontinus *praetor urbanus* and consul, a year later he became commander of Legio II Adiutrix in the campaign against the Batavians. From 74 he was governor *(legatus Augusti pro praetore)* of the province Britannia, until he was replaced by his successor Agricola in 77. Frontinus then served as commander of the province *Germania inferior,* an office for which he was responsible for the construction of the baths in Trier and for the aqueduct Eifel—Köln (total length 91 km(!)). Later he worked as proconsul in the province Asia. In 97 n. Chr. he was appointed c*urator aquarum* of the city of Rome and was responsible for the operation and maintenance of the water supply of the metropolis Rome. The work *De aquae ductu urbis Romae* written by him contains a detailed description of the water-technical facilities of Rome. A complete edition of all Frontinus' works in English is offered by Benett[32].

Frontinus[33] (*De Aqua,* I, 16) is convinced that it is the most important task of architecture to build aqueducts and water pipes—in contrast to the construction of pyramids and Greek temples. His Roman architectural understanding is so different from the Greek:

[32] Frontinus S.J., Benett C.E. (eds.): The Complete Works of Frontinus, Delphi Classics 2015.

[33] Frontinus S.J., Dederich A. (ed.): On the aqueducts of the city of Rome, August Prince 1841.

> With these, so many and necessary water pipes, one may compare the superfluous pyramids or the useless, widely praised buildings of the Greeks.

That this was a general Roman consensus is shown by the quote from Cicero [De officiis II, 60]:

> It is also better to spend on city walls, shipyards, ports, water pipes and other works that contribute to the general good … As for the theaters, colonnades, new temples, I would … express my criticism with restraint; but the most learned men do not approve of such expenses, for example … Demetrius of Phaleros, who criticizes Pericles, one of the first men of Greece, that he has wasted such large sums of money on those magnificent propylaea.

The following are excerpts from Frontinus' work, which deals with arithmetic, as no other work does. A complete table of the nozzles mentioned by Frontinus can be found in Landels[34]. First, he compares the areas of a square with sides 1 *digitus* (finger width) and the inscribed circle. Since 4 finger widths make a hand width, and 4 hand widths make a *foot,* 1 *digitus* $= \frac{29,57}{16}$ cm $= 1,848$ cm. He writes in chapter (I, 24):

> The area of a square with sides 1 *digitus* is greater than the area of the inscribed circle by $\frac{3}{14}$ of its own size; the circle has a smaller area than the square by $\frac{3}{11}$ of its size, since the square corners are missing here.

From this, the value of π he used can be determined: If a is the side length or diameter, then the following must apply

$$a^2 - \frac{3}{14}a^2 = \frac{1}{4}\pi a^2 \therefore \frac{1}{4}\pi a^2 + \frac{3}{44}\pi a^2 = a^2$$

Both equations yield the Archimedean value of $a = 1$ for

$$1 - \frac{3}{14} = \frac{1}{4}\pi \therefore \frac{1}{4}\pi + \frac{3}{44}\pi = 1 \Rightarrow \pi = \frac{22}{7}$$

About 150 years earlier, the architect Vitruv had used the circle constant $\pi = 3\frac{1}{8}$; a value that only reappears in Albrecht Dürer. The constant results from Vitruvius' statement that a wheel with a diameter of 4 [feet] has a circumference of 12½.

In section 1.26 of his book, Frontinus describes four different water pipes that are used in water pipes:

- Digitus *rotundus* has a circular cross-section with a diameter of 1 *digitus.*
- Digitus *quadratus* has a square cross-section with a side of 1 *digitus.*
- *Quinaria* has a circular cross-section of $\frac{5}{4}$ of the *rotundus-diameter.*
- *Uncia* has a circular cross-section with a diameter of 1 *uncia.*

[34] Landels J.G.: Die Technik der antiken Wissenschaften, S. 66, C.H. Beck München, 1989.

The unit 1 ounce *(uncia)* is defined by $\frac{1}{12}$ *feet*$= \frac{29{,}57}{12}$ cm $= 1,848$ cm. It compares the cross section of a *quinaria* tube with that of an *uncia:*

> The 1 ounce tube has a diameter of 1 1/3 digits. Its cross section is more than 1 1/8 *quinariae,* this is 1 and 1/8 of a quinaria tube plus 3/288 and 2/3 of 1/288.

The first claim follows from: $\left(1+\frac{1}{3}\right)\frac{1}{16} = \frac{1}{12}$; the second from

$$\left(\frac{4}{3}\right)^2 \div \left(\frac{5}{4}\right)^2 = \frac{256}{225} = 1\frac{31}{225} > 1\frac{1}{8}$$

The way Frontinus determined the given fractional values is unclear. The deviation is small:

$$1+\frac{1}{8}+\frac{3}{228}+\frac{2}{3}\cdot\frac{1}{288} = 1\frac{119}{864} \therefore \left|1\frac{31}{225} - 1\frac{119}{864}\right| = \frac{1}{21.600}$$

Furthermore, he claims that the cross section of a digitus *rotundus* is 7/12 and 1/24 plus 1/72 of a *quinarias*. However, the error is larger here:

$$\frac{7}{12}+\frac{1}{24}+\frac{1}{72} = \frac{23}{36} \therefore \left|\frac{23}{36} - \frac{16}{25}\right| = \frac{1}{900}$$

F) Measuring a river width

Determining a river width, if the opposite bank is not accessible or occupied by the enemy, was a typical task for land surveyors.

M.Cantor also mentions the land surveyor Sextus Julius Africanus, who came from today's Libya (around 200 AD); although he writes in Greek, he should be mentioned here. In his work κεστοι (XXXI) he describes a method for measuring the width of a river (Fig. 25.16a). A salient point *C* on the opposite bank is sighted

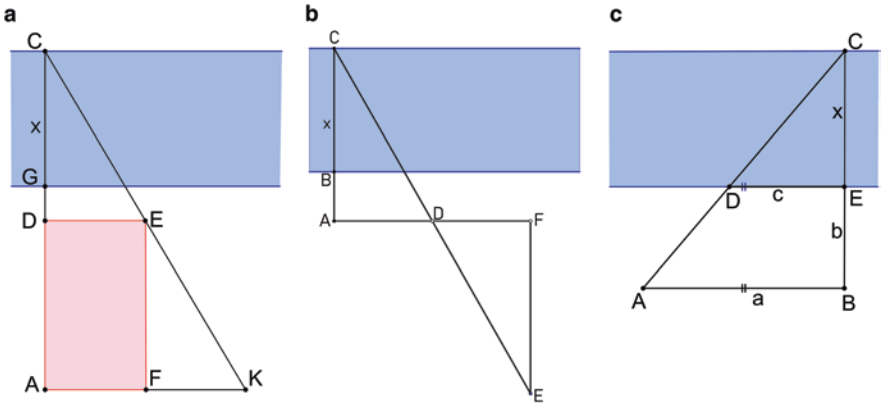


Abb. 25.16 Methods for measuring a river width

from a point A so that the line AC forms a perpendicular line to the river bank. The distance AK is laid out parallel to the river bank and the midpoint F is marked. From the endpoint K the point C on the opposite bank is sighted again. Between the sight lines AC and KC the parallel distance $|DE|$ is fitted with $|DE| = |AF|$. So that the rectangle $AFED$ is determined. The resulting triangle $\triangle EFK$ is congruent to $\triangle CDE$; thus $|EF| = |CD|$. Subtracting the distance $|DG|$ from $|CD|$ yields the desired river width x.

Also simple is the method described by Frontinus (Fig. 25.16b). Point A is chosen such that the sight line to a point C on the opposite bank is perpendicular to the river bank. At A, the line AD is marked parallel to the shore and produced by the same length $|AF| = |AD|$. The perpendicular to AF at point F is erected away from the shore far enough so that one can see from endpoint E the point C on the other side exactly over D. Triangle $\triangle DEF$ is congruent to $\triangle ADC$; thus $|AC| = |EF|$. Subtracting from $|EF|$ the distance $|AB|$, one obtains the desired river width x.

The more elegant method using similarity (Fig. 25.16c) was not known. At the riverbank, the lines DE and AB parallel to it are laid out in such a way that the sight line runs over B, E to the opposite point C perpendicular to the riverbank. The endpoints A, D should also lie on a line of sight to point C. Then the lines $|AB| = a, |BE| = b, |DE| = c$ are measured. Because of the similarity of the triangles $\triangle ABC$ and $\triangle DEC$, the desired width of the river x results in

$$\frac{a}{c} = \frac{b+x}{x} \Rightarrow x = \frac{cb}{a-c}$$

The method is also suitable for measuring the wall height of a besieged city by means of a sighting rod. The sighting method is illustrated in Fig. 25.17; in the lower part you can see the *chorbates* instrument described by *Vitruvius* (VIII, 5.1), which was also suitable for measuring inclination.

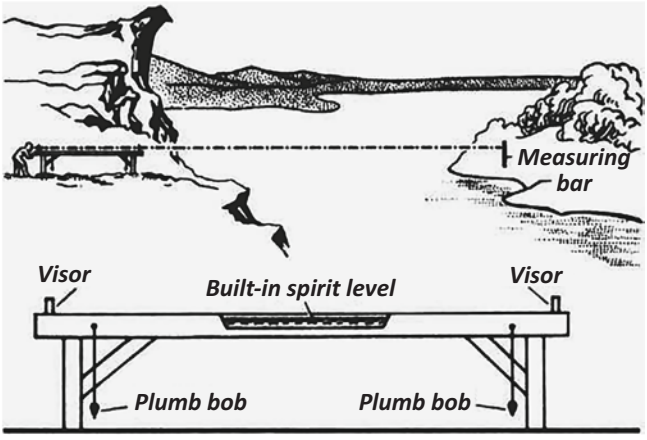


Fig. 25.17 Surveying device Chrobates (after Vitruvius) and application in the field, Lexikon der Technik, Leipzig 1980

E) Further constructions of the surveyors
Another task of the surveyors was to determine the east-west direction (Fig. 25.18a). Hyginus describes a simple method for this. For this purpose, a circle was drawn around a (vertically standing) gnomon, the radius of which was smaller than the longest shadow length. If you mark the point at which the shadow just touches the circle in the morning and afternoon, you will get the east-west line (*decumanus* maximus) by connecting the two points.

Fig. 25.18b shows the construction of a right angle, which is not found in Euclid. On the given line, the distance AB is arbitrarily marked and in the end-points two circles are drawn each with radius $|AB|$. The intersection point C, which forms an equilateral triangle with A, B, is the center of another circle with radius $|AC| = |BC|$. The line BC intersects the circle at C at the point D, which determines the perpendicular line AD. With the help of this perpendicular construction, the vertical line *(kardo maximus)* could be constructed from the east-west line.

The overall assessment of J.L. Heibergs on Roman mathematics is somewhat polemic:

> The overall impression of Roman *art of surveying* is that it is 1000 years older than Greek geometry and the deluge happened in the meantime.

Epilogue On the fate of Aeneas from the Trojan legends, Publius Vergilius Maro *(*Vergil*)* tried to make the whole heroic story of the Roman people visible. In his opinion, the Greeks exceed the Romans in skillfulness, but the Romans will be the masters of the world, proud and arrogant:

> Others, I believe, will beat out bronze that seems to breathe and chisel living faces out of marble.They'll excel in pleading lawsuits, and they'll trace the heavens' paths and chart the rising stars.You, Roman, remember your own arts: to rule the world with law, impose your ways on peace, grant the conquered clemency, and crush the proud in war.
> (Vergil, Aeneïd VI, 847 ff)

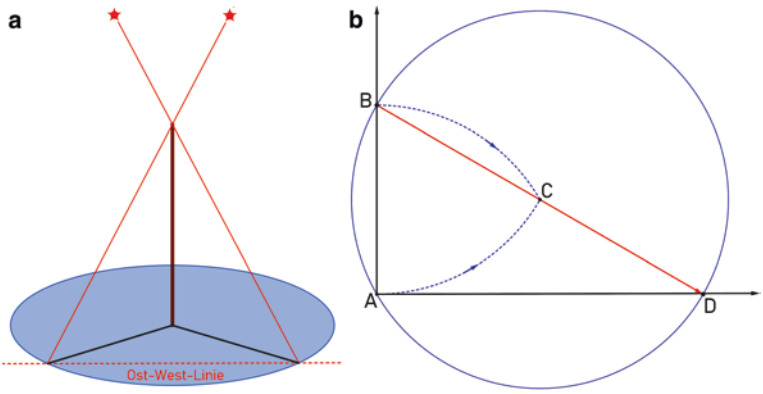


Fig. 25.18 Two constructions of surveyors

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The Legacy of Hellenistic Mathematics

26

26.1 Mathematics in Byzantium

The Neoplatonic Academy in Athens was closed in 529 by Emperor Justinian I (reign 527–565). Two of the remaining teachers *Simplicius* and *Damascus* (from Damascus) went to the court of the Persian king Chosroes I with others, but left after only one year. All but one returned to Athens. The one who went to Constantinople was Isidore of Miletus, who later became one of the architects of the famous *Hagia Sophia* (*Holy Wisdom*). The two architects of the Hagia Sophia (after the collapse caused by the Nika riot) were Anthemius of Tralleis and *Isidore of Miletus*, with the latter taking over the construction management after the death of the former (re-constructed in 532–537 AD). Constantinople was the old city of Byzantium, which was founded by Constantine in 324 and named after him. Isidore was also the head of a school where Euclid's *Elements* were taught. The knowledge gained there was added as the 15th book of the *Elements* later. The author of this book actually names the famous "Isidore" as his teacher.

Anthemios was friends with the mathematician Eutokios from Askalon, who was a student of the last leader of the Alexandrian school *Ammonios*. We owe an important commentary to Eutokios on Archimedes and the first four books of *Conica* by Apollonius.

Leon the Geometer (from Byzantium), also called *Leon the Philosopher*, collected numerous mathematical manuscripts by Archimedes, Apollonius, Proclus and Theon of Alexandria and had copies made of them; he wrote four epigrams about them, which are preserved in the *Anthologia Graeca* (IX, 200, 202, 203, 578). Epigram 202 celebrates the latter two:

> Greetings [be] to you, Master Theon, who knows everything
> and crowns Alexander's city with your achievement

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> Greetings also to you, Proclus, who as the best offspring
> of Sarpedon[1] you are praised by all the world!

The most famous Euclid manuscript is now in Oxford; an Archimedes copy came via Norman ownership to the Vatican, where it disappeared in the 16th century. The most famous Ptolemy manuscript also comes from Constantinople. During one of the many sieges of Constantinople by the Arabs, a student of Leon fell into captivity and reported at the court of the Caliph *al-Mamun* about the translations of Leon. However, Leon declined the invitation of the Caliph. The Eastern Roman Emperor Theophilus (ruled 829–842) rewarded Leon's stay by appointing him head of the new school for philosophy and science.

Thus, Constantinople preserved important parts of the Hellenistic heritage. This succeeded until the city on the Bosporus was conquered by Ottoman troops after a long siege in May 1453. The last ambassador of Rome was none other than the famous Christian scholar *Cusanus* (Nicolas of Cusa). Rome and the Republic Venice had not been willing to support the Christian population of Constantinople sufficiently with troops. It must have been a great triumph for Sultan Mehmet II to enter the Hagia Sophia on the day of capitulation. Since the separation of the Roman empire in 395 AD into a Western and Eastern part, there had been considerable tension between the two parts. Then, in 1054, there was a split into the Roman Catholic and Greek-Orthodox churches, respectively. As a revenge for the year 1189, when Constantinople refused passage to the army of Barbarossa, the 4thCrusade sacked the city completely in 1024. Fig. 26.1 shows the Hippodrome of Byzantium, the counterpart to the Circus Maximus in Rome, after the Islamic conquest. The most famous war booty was the bronze Quadriga (four-horse chariot) of the Hippodrome and the Four Tetrarchs group, which today are in St. Mark's Cathedral (Venice).

The most important (Greek-writing) Byzantine scholar was M. Planudes (1255–1310), who wrote commentaries on the first two books of Diophantus. Furthermore, he is also the author of the work *Calculating in the manner of the Indians,* which was handed down together with the *Anthologia Graeca*. Planudes was the ambassador of the Emperor Andronikus II to the Republic of Venice in 1297 and had probably studied there a manuscript from 1225 by an unknown author on Indian calculation (Ινδικης ψηφιφοριας). In Western Europe, the Indian numbers already appeared in the book *Liber Abbaci* (1202) by Leonardo von Pisa.

26.1.1 From the Greek Anthology (Anthologia Graeca)

The Anthologia originated around 70 BC from a collection of epigrams by the poets Meleager, Philip, and Agathias, later extended by a collection of erotic poems by Straton of Sardis (being added in the 2nd century AD). The clergy and

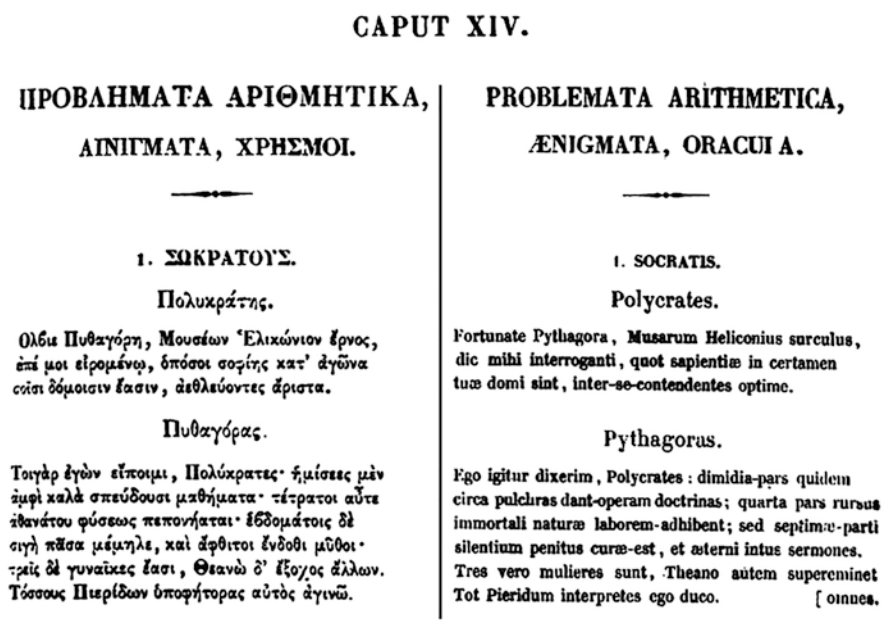
[1] King of Lycia, Der kleine Pauly, dtv 1979, column 1559.

Fig. 26.1 The grand vizier visits the Hippodrome in Constantinople, painting by Vanmour (around 1710), AKG585462 Copyright / akg-images

man of letters Kephalas added, in the 10th century, the epigrams of the poets or authors Callimachus, Theocritus, Diogenes Laertius, Christian poems, and pagan temple and tomb inscriptions, thus collecting the 15 books of the *Anthologia Graeca*.

The aforementioned Byzantine scholar *Maximus Planudes* produced his own edition of the Anthology in 1301, editing and adding various texts to numerous poems which today make up Book XVI. Book XIV contains approximately 140 riddles and problems, which were mainly compiled by the grammarian *Metrodoros* (c. 520 AD). Together with the Anthology, the Planudes's arithmetic book has also been handed down. A manuscript by Planudes was found in San Marco (Venice) and first printed in 1494. The Anthology also received the name *Palatina* (Latin: Palatinate) after the library of the same name, from which it came to Rome and Paris after the French conquest of Heidelberg. Fig. 26.2 shows the beginning of Chap. XIV of a Greek-Latin edition.

Book XIV contains, in addition to various puzzles, 46 exactly arithmetic problems. Of these, 23 are linear equations, like the famous problem (No. 126) about the age of Diophantus. Twelve problems contain systems of linear equations with two unknowns. Six problems are of the type "tube problem". The two problems (No. 48, 144) provide linear, indeterminate equations that go beyond Diophantus. The three remaining problems (No. 51, 49, 144) include linear systems with three or four unknowns and a parameter solution.



CAPUT XIV.

ΠΡΟΒΛΗΜΑΤΑ ΑΡΙΘΜΗΤΙΚΑ,
ΑΙΝΙΓΜΑΤΑ, ΧΡΗΣΜΟΙ.

1. ΣΩΚΡΑΤΟΥΣ.

Πολυκράτης.

Ὀλβιε Πυθαγόρη, Μουσέων Ἑλικώνιον ἔρνος,
ἐπί μοι εἰρομένῳ, ὁπόσοι σοφίης κατ' ἀγῶνα
σοῖσι δόμοισιν ἔασιν, ἀεθλεύοντες ἄριστα.

Πυθαγόρας.

Τοιγὰρ ἐγὼν εἴποιμι, Πολύκρατες· ἡμίσεες μὲν
ἀμφὶ καλὰ σπεύδουσι μαθήματα· τέτρατοι αὖτε
ἀθανάτου φύσεως πεπονήαται· ἑβδομάτοις δὲ
σιγὴ πᾶσα μέμηλε, καὶ ἄφθιτοι ἔνδοθι μῦθοι·
τρεῖς δὲ γυναῖκες ἔασι, Θεανὼ δ' ἔξοχος ἄλλων.
Τόσσους Πιερίδων ὑποφήτορας αὐτὸς ἀγινῶ.

PROBLEMATA ARITHMETICA,
ÆNIGMATA, ORACULA.

1. SOCRATIS.

Polycrates.

Fortunate Pythagora, Musarum Heliconius surculus,
dic mihi interroganti, quot sapientiæ in certamen
tuæ domi sint, inter-se-contendentes optime.

Pythagoras.

Ego igitur dixerim, Polycrates: dimidia-pars quidem
circa pulchras dant-operam doctrinas; quarta pars rursus
immortali naturæ laborem-adhibent; sed septimæ-parti
silentium penitus curæ-est, et æterni intus sermones.
Tres vero mulieres sunt, Theano autem supereminet
Tot Pieridum interpretes ego duco. [omnes.

Fig. 26.2 Book XIV of a Greek-Latin edition of the Anthologia Graeca

1) Blessed Pythagoras, offspring of the Heliconian [2] Muses, please give me information: How many students in your house are engaged in the contest for wisdom performing excellently?

I am happy to answer, Polykrates: The half of them study the noble mathematics diligently. $\frac{1}{4}$apply themselves to studying immortal nature. $\frac{1}{7}$ is diligent in strict silence, never destroys values in the heart. In addition, 3 women are part of it, as the best *Theano.* So many young Pierian girls [the Muses] I educate today! [Solution: 28]

Note: The mentioned Theano is one of the few female students of Pythagoras, who are known by name; her name is also mentioned by Iamblichos.

3) Eros is dejected. Then Kypris asked him: What kind of trouble torments you, young man? He gave her the answer: The Pierides [Muses] snatched the apples from me that I was just fetching from Helicon and then quickly dispersed like the wind. Klio snatched $\frac{1}{5}$ of the apples, Euterpe $\frac{1}{12}$, but a $\frac{1}{8}$ the glorious Thalia; and Melpomene snatched $\frac{1}{20}$ away, Terpsichore, on the other hand, $\frac{1}{4}$; Muse Erato took $\frac{1}{7}$ from the loot, 30 from the supply Polyhymnia abducted, 120 Urania stole; and Calliope loaded herself with a burden of 300 and, heavy-footed, sought her escape. On my part, I come to you with powerfully relieved arms, bringing only 50 apples that the goddesses allowed me. [Solution: 3360]

[2] Mountains in Boeotia, The Little Pauly, dtv, column 994.

7) I am a bronze lion. Water spurts out of my eyes, my mouth, and also from my right paw into the basin. My right eye takes 2 days to fill the basin, but the left one only 3, finally my paw 4. Of course, my mouth only needs 6 h. Now tell me: How long does it take for all of them together, eye, paw, and mouth? (1 day = 12 h) [Solution: $3\frac{33}{37}$ h].

12) Kroisos (Croesus), the king, dedicated 6 bowls that weighed 6 mines, each bowl is one drachma heavier than the previous one weighed. (1 mine = 100 drachmas). [Solution: $\left(97\frac{1}{2}; \ldots; 102\frac{1}{2}\right)$]

49) Forge me a wreath! (Mixture problem, see Sec. 4.8)

51) I weigh as much as my neighbor plus $\frac{1}{3}$ of the third person. I weigh as much as the third person plus $\frac{1}{3}$ of the first person. 10 mines I, in addition $\frac{1}{3}$ from the second. [Solution: 45; $37\frac{1}{2}$; $22\frac{1}{2}$]

123) Take $\frac{1}{5}$, my son, from my property; $\frac{1}{12}$ come, wife, to you; you, the 4 children of the son who already died, you 2 brothers, you, painfully groaning mother, each take $\frac{1}{11}$ as an inheritance share. You, as cousins, you should each receive 12 talents; Eubulos, as a friend, 5 of the talents are granted. I give freedom and compensation to my most faithful slaves; I pay them for their services as follows: Onesimus may have 25 mines, Davos take 20 in possession as wages, 50 my Syros, Tibios 8 and 10 Synete, while Syros' son, Synetos, 7 receives. You may use 30 talents for the maintenance of my grave, and also for the sacrifice to Zeus who reigns in the underworld; 2 for the funeral pyre, the funeral meal, the shrouding of the body; 2 more for further service to the perishable body. (1 Talent = 60 mines) [Solution: 660]

126) Age of Diophantus (see Chap. 21).

136) As a bricklayer, I would like to speed up the construction of the house! Today the sky is cloudless, I only need a few bricks, because I only lack 300 in total. You alone burn that many in one day. Of course, your offspring stops working after 200, while your son-in-law at least manages 50 more. How many hours do the three of you need for joint work? [Solution: $\frac{24}{5}$ h]

144) "My weight increases considerably together with the base." "I'm sure I weigh the same as my base." "But I weigh twice as much as your base alone." "And I only have three times your base weight." [Conversation of two statues x, y with base x_1, y_1; Parameter solution: $(x, y, x_1, y_1) = \left(2t; \frac{3}{2}t; \frac{1}{2}t; t\right)$]

145) Give me 10 mines, then I'm three times as heavy as you. Give me the same, five times I'm then as heavy as you. [Solution: $15\frac{5}{7}$; $18\frac{4}{7}$]

The well-known donkey problem, which is attributed to Euclid, should be added here. Philipp *Melanchthon* liked it so much that he translated it into Latin.[3]

A donkey laden with a heavy burden went and the donkey's mother
and the donkey sighed heavily, then the little son said:
Mother, why do you lament and groan like a grieving maid;
Give me a pound, then I'll carry the double burden,
But if you take it from me, we'll both have the same.
Calculate for me, if you can, my best, how much she carried!

[3] Euclidis opera omnia, Ed. Heiberg und Menge, Band VIII, S. 286.

26.2 Outlook on the Latin Early Middle Ages

As a result of the Migration of People, there were numerous wars from the 4th century onwards as the barbarians invaded the Western Roman Empire; it was not able to withstand the onslaught of the Goths, Vandals and Franks. With the breakdown of the Western Roman Empire, the Roman education system, which was largely based on Greek educational values, came to an end. The philosophical writings of the ancients were rejected as pagan, in particular the astronomical works were used for astrology, magic and superstition.

The most prominent and important cultural asset inherited from antiquity was the Latin language, which for a long time established itself as the universal European language of religion, culture, and science and was closely associated with the ideology it conveyed. Only later did people try to shake off the spiritual tutelage of the church and communicate in the respective national languages. An example of this is the publication of the first German-language Bible, which was printed in 1466 by Johannes Mentelin in Strasbourg.

Few impulses for a natural science education come from the early church teachers *Tertullian* (160–220 AD), *Origines* (185–254) and *Augustinus* (354–430), who all still went through the ancient educational system. Ostrogothic king Theoderic (453–526) gathered two scholars of old Roman nobility at his court as officials and secretaries in Ravenna. Flavius Aurelius *Cassiodorus* (490–583) and Manlius *Boethius* (480–526) tried to bring the old pagan teachings into harmony with the new Christian doctrine. There was a long standstill in the sciences; one merely lived off the knowledge that the encyclopedists had gathered from old writings. Fig. 26.3 shows Cassiodorus standing to the right next to the personified *Geometria*.

The most important encyclopedist of the early Middle Ages was the West Gothic bishop Isidore of Seville (560–636), who contributed to keeping the scientific knowledge of the Greeks alive in the Latin West. His etymology, which was based on often fantastic derivations of various *Termini technici*, remained a source of much knowledge about astronomy and medicine for many centuries. For Isidore, the universe was limited in size, only a few thousand years old and close to destruction. He thought that the earth had the shape of a disk surrounded by the ocean. Around the earth were the concentric spheres that carried the planets and stars, and beyond the last sphere was the highest heaven. This is beautifully illustrated in the well-known (colored) woodcut by E. Flammarion in Fig. 26.4; the original black and white version of the image was long thought to date from the Middle Ages until it was discovered to be the work of Flammarion (1888).

Other encyclopedists were Bede Venerabilis (675–735), Alcuin of York (735–804) and Hrabanus Maurus (776–856). Each of them copied without worrying about their predecessors. That so much was preserved was due to the foundation of monasteries with the schools attached to them. This development began in Western Europe after the foundation of Monte Cassino by Benedict of Nursia in 529.

The existence of such educational centers made it possible for a temporary revival of science, in the 6th and 7th centuries in Ireland, in Northumbria

Fig. 26.3 Cassiodorus and Geometria, manuscript of the Bayerische Staatsbibliothek

Fig. 26.4 Woodcut by Flammarion (1888), colored, (Wikimedia Commons)

during Bede's time and in the 9th century in the empire of Charlemagne. The latter invited Alcuin from Northumbria to his court and made him his Minister of Education. One of the most important reforms Alcuin carried out was the

foundation of monastery and cathedral schools. In these institutions, the writings of the encyclopedists determined the curriculum until the 12th century. The study was limited to the seven liberal arts, as defined by Varro in the 1st century BC and later by Boethius. Grammar, logic and rhetoric formed the first stage or the *Trivium* and geometry, arithmetic, astronomy and music the more advanced *Quadrivium*. The works of Pliny, Boethius, Cassiodorus and Isidore (of Seville) served as textbooks. With the Christianization, a significant development took place, namely the assimilation of Neoplatonism.

This was of great importance, because it determined the world view until the middle of the 12th century. Through Augustine (of Hippo) (354–430) fundamental topics of Greek philosophy flowed into the thinking of the Christian Occident; Augustine himself was influenced by Plato and Neoplatonics like Plotinus (around 203–270 AD). In the field of natural sciences, the ideas of Aristotle were adopted, insofar as they fitted into the Christian world view. These were in particular:

- The geocentric world view.
- God as the "first unmoved" mover of the heavenly spheres.
- All heavenly bodies move in fixed orbits.
- Explanation of free fall: All things have their natural place.

Augustine [De Libero Arbitro II, 8, 21] took over the belief in the indestructible truth of numbers:

> And I do not know how long anything that I touch with a physical sense organ will last, such as this heaven and this earth and whatever else I perceive as other bodies in them. But 7 and 3 are 10, and not only now, but always; 7 and 3 have also never been not 10 at any time, nor will 7 and 3 ever not be 10 at any time. Therefore, I said that this indestructible truth of numbers is general, for me and for anyone who thinks at all.

However, astronomy, like all general science, must give way to theology. In the *De Genesi adversus Manichi* , Augustine stated:

> In the Gospel cannot be read that the Lord said: I send you the Holy Spirit so that he may instruct you about the course of the sun and the moon. He wanted to make Christians, not mathematicians.

By "mathematics" Augustine still means astrology. This explains a *curious* passage in the *City of God (V, 3)*. Nigidius, a contemporary of Cicero, had made the following attempt to explain why twins born shortly after each other (according to the astrologers) did not experience the same fate. He let a ball rotate on a potter's wheel and quickly marked two points that were far apart after the ball had stopped. This was supposed to serve as a metaphor for the different star constellations and thus also the life paths of twins. Augustine writes about this (V, 4):

> But if one relies on the fact that here the statements do not rest on such short time intervals that they escape perception …, what then is the metaphor of the potter's wheel, which only drives people with earthly hearts around in a circle, *so that the deception that the mathematicians practice* does not come to light.

Fig. 26.5 Boethius and Pythagoras, manuscript Cambridge (11th century), Science Photo Library

26.2.1 Boethius

The Ostrogothic king *Theoderich* had violently seized power from the Germanic mercenary leader Odoacer, who in 476 had followed the last Western Roman emperor Romulus Augustulus, and had to deal with the ruling Roman nobility, who considered the Goths residing in Ravenna as foreign rulers. One of the numerous Roman officials at the court was Anicius Manlius Severinus Boethius (480–524), who was 485 Roman consul and also a versatile scholar. His stepfather and father-in-law Quintus Aurelius *Symmachus* had retired from politics and become a writer. His Roman History in 7 volumes has not survived; however, some quotations can be found in the *Getica* of Jordanes. When the Burgundian king Gunibald wanted a water and sun clock, Theoderic commissioned Boethius to build the clocks. In a letter preserved by Cassiodorus, Theoderic wrote to Boethius:

> In your translations [from Greek into Latin] the music of Pythagoras, the astronomy of Ptolemy will be read in Latin, Nicomachus the arithmetic and the geometry Euclid will be … heard. You have also given us Archimedes, the mechanic, in Latin. … Rome received them in the mother tongue through your only mediation.

Cassiodorus noted in his anthology *Variae*: *The subject of arithmetic was once carefully presented among the Greeks by Nicomachus. The first to introduce it to the Romans was Apuleius of Madaura and then the famous man Boethius by translating it into Latin*. Fig. 26.5 shows Boethius sitting on the left with a monochord on his lap, next to him on the right Pythagoras, who is striking a glockenspiel (chimes) with a hammer and holding a set of scales with hammers in one hand. Boethius, who remained loyal to his employers, was accused of being a member of an (Eastern Roman) conspiracy against Theoderic. During his long imprisonment,

he wrote his most famous philosophical-poetic work *Consolatio philosophiae.* A famous quote from it that is still relevant today (II, 7):

> You have learned from astronomical evidence that the whole earth, compared to the universe, is no larger than a point, that is, compared to the sphere of the heavens, it has virtually no extension. Of this tiny corner, only a quarter is habitable according to Ptolemy. If you subtract the seas, swamps, and other desert areas from this quarter, the space that remains for man is hardly worth being called infinitesimal.

He was then executed in 524, as was later his father-in-law.. The most important writing that Boethius translated from Greek into Latin was the work[4] *De institutione arithmetica* by Nicomachus. He himself writes in the preface:

> Since I do not have to follow the plans of others, I have taken the liberty of taking my own path and not following the footsteps of others. What Nicomachus gives in more detail about numbers, I have summarized briefly. What was run through quickly and only allowed limited access, I have clarified by moderate addition … That this [approach] has cost us many nights of sleep and sweat, a sensible reader can easily see.

His concept of numbers is simplified compared to Nicomachus; while the latter still distinguished three aspects of the concept of numbers, Boethius only knows two: *Numerus est untitatis collectio vel quantitatis acervus unitatibus profusus* (I, 3). A number is a collection of units or a set of quantities originating from the units. He takes over all definitions, such as even or odd, prime or composite number, from Nicomachus. He uses Roman numerals, writing 4 as IIII.

The term *quadruvium* (crossroads, later *quadrivium*) for the mathematical disciplines of arithmetic, geometry, music and astronomy can be found for the first time in the *Arithmetica*. For the operation of teaching in monasteries and universities, this division of knowledge into seven liberal arts became decisive. In his introduction, Boethius emphasizes the importance of the quadrivium:

> It is considered as given by all men of great reputation who adhere to the Platonic doctrine and are distinguished by their intellect, that none can achieve the highest perfection in philosophy who has not gone the way of the quadrivium.

Next to the quadrivium, the *trivium* (triple path from grammar, rhetoric and dialectic) was taught until the Renaissance and Reformation.

In addition to arithmetic, the preserved manuscripts also contain the so-called *geometry* of "Boethius" (Pseudo-Boethius). As M. Folkerts[5] has shown, this was created in the 11th century as a summary of mainly two writings. One is the selection originally ascribed to Boethius from the *Elements:* Postulates, Axioms and Propositions from Book I, the first ten Propositions from Book II, some from

[4] Anicii Manlii Torquati Severini Boetii: De Institutione Arithmetica, Friedlein G. (ed.), Leipzig 1867.

[5] Folkerts M.: "Boethius" Geometry II, Franz Steiner 1970.

Fig. 26.6 Isidore of Seville and Bishop Braulio, Monastery of Einsiedeln (Wikimedia Commons)

Book III and IV. The other writing of Pseudo-Boethius contains the abacus manual by Gerbert of Aurillac, who later became Pope Sylvester II in Avignon. For reasons of scope, it cannot be discussed here.

26.2.2 Isidore of Seville

Isidore of Seville (lat. *Isidorus Hispanis*) (ca. 560-636) was born in Cartagena (southern Spain) and, like his siblings, chose a church career. Fig. 26.6 shows Isidore sitting to the right next to his bishop Braulio (with bishop's bar). Isidore's birthplace was part of the Visigotic Kingdom, which was formed as a result of the migration. The West Goths had come to northern Italy as allies of the Eastern Roman Empire in the 5th century, had plundered Rome after becoming weaker of the Western Empire in 410, and had been driven out of Italy after the death of their famous king Alaric. They first moved to southern France (412), where they were settled by the Eastern Roman Empire in Spain together with the Suebi.

In Isidore's youth, the Visigothic king Leovigild united the Iberian Peninsula under his power. The conversion of the king's son Rekarred from Arianism to Catholicism also brought religious peace to the land. Under the kings, the bishops became politically more powerful, as the councils served as legislative assemblies. Training centers of the Visigothic Empire were the three bishoprics of Toledo, Zaragoza and Seville. Probably on behalf of this king, Isidore wrote a number of works in Latin, including the *Encyclopedia.* Therefore, some historians see Isidore as the last author writing Latin as his mother tongue.

Isidore's *Encyclopedia*

The 20-volume Encyclopedia[6] was the most widely circulated manuscript in the Middle Ages; over 1000 copies of this *handbook* are known from all cultural centers of the Western world. The manuscript is quoted by all later church scholars, including Bede Venerabilis and Hrabanus Maurus, the famous pupil of Alcuin, who is called the *Teacher of Germany* in English literature. The Latin source text[7] of the *Encyclopedia* was not published until 1911 by W. M. Lindsay. Ernst R. Curtius[8], the author of the fundamental work *European Literature of the Middle Ages* calls it the *basic book of the Middle Ages.*

The work served a threefold purpose. First and foremost, it was a wordbook or etymology (Greek ἔτυμολογιας with the word stem ἔτυμον, true word sense), as the title of the English translation[9] suggests. Furthermore, it served as a concise outline of all knowledge. Finally, it provides an overview of the world view that consisted of handed-down Greek-pagan elements and Christian doctrine. The pagan writings were no longer read, because there were only a few scholars who understood Latin and Greek. Gregory of Tours (538–594) complains:

> The misfortune of our days is that the study of the old writings is dead, there is no one who preserves the history of the old times.

Isidore of Pelusium, first a rhetorician, later a monk in the desert, writes around 420:

> The faith of the Greeks, which for so many years set the tone and caused such pain, spent so much money, waved their weapons so often, has disappeared from the earth.

Fredegar Scholasticus regrets in his chronicle (around 640):

> The [mental] world is dying; intellectual life is extinguished and the old writers have found no successors.

However, the biographer of St. Elegius doubts the sense of old education:

> What are we supposed to do with the so-called philosophies of Pythagoras, Socrates, Plato and Aristotle or with the trash and nonsense of such shameless poets like Homer, Vergil and Menander? What sense does it make to read the writings of Sallust, Herodotus, Livius, Demosthenes or Cicero for our servants of God?

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[6] Isidore of Seville, Möller L. (ed.): The Encyclopedia of Isidore of Seville, Marix 2009.

[7] Lindsay W. M.: Isidori Hispalensis Episcopi Etymologiarum sive Originum Libri XX, Oxford 1911.

[8] Curtius E. R.: Europäische Literatur und Lateinisches Mittelalter, Bern 1948.

[9] Barney S. A., Lewis W. J., Beach J. A., et al. (Ed.): The Etymologies of Isidore of Seville, Cambridge 2006.

The mathematical knowledge of the encyclopedia

> *Tolle numerum in rebus omnibus, et omnia pereunt.*
> (Take the number from all things and everything perishes, Book III, 4, 4)

Isidore devotes Book III to mathematics in the wider sense:

> Mathematics has 4 subdisciplines: arithmetic, music, geometry and astronomy. Arithmetic is the science of the countable quantity. Music is the science of the numbers that can be found in sounds. Geometry is the science of sizes and shapes. Astronomy is the science of the movement of heavenly bodies and the positions and images of stars.

Isidore takes over the definitions of his role models *Cassiodorus* and *Boethius.* Cassiodorus (485–580) was a senator of old Roman nobility who served the Ostrogothic king Theoderic the Great and, after his death (526), took over the state affairs for Theoderic's daughter. Finally, after the end of the Ostrogothic Empire, he founded the first monastery in the Western Roman Empire and supplied it with numerous own writings. He adopted the theory of means from Nicomachus and his commentator Iamblichus. The difference between arithmetic, geometry and music lies in the choice of means. Isidore explains the arithmetic mean of two numbers with the example of 6 and 12. Added together, 18 results, divided in the middle 9 results; the difference of both numbers to the mean $12 - 9 = 9 - 6$ is equal. In general, the arithmetic mean A of two numbers $a < b$ has the same distance to both numbers

$$A - a = b - A \Rightarrow A = \frac{a+b}{2}$$

Looking for the geometric mean of $a = 6; b = 12$, one forms the product of the numbers $6 \times 12 = 72$; this is equal to the product of the means $H = 8$ and $A = 9$. Here Isidor falsely equates the geometric mean with the product of arithmetic and harmonic mean. In fact, this product is equal to the square of the geometric mean

$$\frac{a+b}{2} \times \frac{2ab}{a+b} = ab = \left(\sqrt{ab}\right)^2$$

The following description of the harmonic mean is incomprehensible, the text of the manuscript seems to be corrupted here. A correct description of the harmonic mean—as an application of music—can be found in book (3, 23). He explains this again with the numerical example $a = 6; b = 12$: The difference of the numbers is 6, the square 36, the sum makes 18. 36 divided by 18 results in 2. If you add two to the smaller number, you get 8; 8 is therefore the (harmonic) mean of 6 and 12. 8 exceeds 6 by 2, that is, by a third of 6. 8 is exceeded by 12 by 4, also by a third of the number. Thus, the larger number exceeds the mean in the same proportion as the mean exceeds the smaller number. If H is the harmonic mean of two numbers $a < b$„ then it actually applies

$$\frac{H-a}{a} = \frac{b-H}{b} \Rightarrow H = \frac{2ab}{a+b}$$

Furthermore, Isidore notes that the number sequence is infinite. Not only can 1 be added to an imagined "end number" at any time, but also any multiple of this number.

Outlook on another encyclopedia
Another important encyclopedia also emerged in the religious sphere. It is the famous (completed in 1496) *Margarita Philosophica* (Pearl of Wisdom) of the Carthusian monk Gregorius Reisch (1467–1525), which features a number of popular woodcuts, among others attributed to Michael Wolgemut (the teacher of A. Dürer). Famous students of G. Reisch were the theologian Johann Eck and the cartographer Martin Waldseemüller. Erasmus of Rotterdam said of Reich: *his teaching in Germany had the weight of an Oracle.*

26.3 Greek legacy in Islam

The truth must also be sought among the distant nations that do not speak our language.
Al-Kindi Ishaq, Ivy A.(Ed.): On the First Philosophy, State University of New York Press, 1974

After the death of the Prophet and religious founder Mohammed in the year 634, the expansion phase of Islam began relatively quickly. In 635 Damascus was conquered, in 637 Persia, in 642 Alexandria, and in 711 the general Tarik crossed over from Gibraltar to Spain.

Nestorians, who were expelled from Constantinople in 489, migrated to Persia, specifically to the school at Gundischapur, famous for its healing arts; many of the 529 scholars affected by the closure of the (new) Academy in Athens had the same destination. In the areas occupied by the Muslims, there were several centers of learning, including in Iraq, Syria, and Egypt, with a branch of the Museion being relocated to Syria. In Harran, located in northern Mesopotamia, there was a religious school of the sect of the Sabians. Since it was strictly forbidden by religious rules to translate the Koran into another language, all people of the conquered countries were forced to learn Arabic, and the Sharia legal system was introduced.

Caliph al-Mansur (Caliphate 754–775) had founded Baghdad as the new capital of the empire after the victory of the Abbasids over the Umayyads in 762. Since there was a great need for doctors, he repeatedly brought physicians from Harran to his court. There was also contact with India in 771, when a Hindu delegation presented the Caliph with their astronomical table work *Sindhind* (in Sanskrit *Siddhanta*). With the translation into Arabic came the use of Indian numerals to Baghdad.

The Caliph *Harun ar-Rashid* (the Rightly Guided One) (Caliphate 786–809) was the grandson of al-Mansur, whose court and wealth are known from *Stories from One Thousand and One Nights* and by the sending of an elephant to Charlemagne. He was a despotic ruler who was able to extort significant tribute payments from Constantinople during the regency of Empress Irene. When her successor, Emperor Nikephoros I, refused the tribute, his troops were defeated

at Phrygia in 805 and he was forced to pay an annual tribute of 30,000 (!) old numismata . At the same time, however, he promoted science and had a library built that contained books in Sanskrit, Old Syriac, Persian, and Greek. It therefore became necessary to translate the existing literature into Arabic; the scholar *al-Ḥaǧǧāǧ* was the first translator of Euclid's *Elements*.

The following Caliph al-Mamun (the Trustworthy) (Caliphate 813–833), son of ar-Rashid, erected the *House of Wisdom (bayt al-hikma*) in Baghdad, which also included a library and an observatory. Al-Mamun extorted numerous manuscripts from Emperor Michael II, whom he had defeated in 823. Many scientists from different regions worked at this institutions - for over 200 years one of the most important science centers, together with Samarkand, Cairo and Damascus. Among them were the mathematician Mohammed *al-Khwārizmī* from today's Uzbekistan, the mathematician and translator Thābit *ibn Qurra* from today's Diyarbakir (Turkey) and the translators Qusṭā ibn Lūqā (from Syria) and Isḥaq *ibn Ḥunayn.* (a former Nestorian Christian). In addition to these scholars, there were also wealthy citizens, like the three brothers Ahmad, Hasan and Muhammed Musa, called Banū Mūsā (Fig. 26.8), who bought manuscripts and either translated them themselves or commissioned other translators. The University of Al Mustansiriya (Fig. 26.7) sees itself as the successor of the House of Wisdom.

The difficulties in translating are shown by way of example with the *Conica* of Apollonius. First, the Banū Mūsā brothers were only able to acquire an incomplete

Fig. 26.7 University of Al-Mustansiriya Baghdad, (Wikimedia Commons)

Fig. 26.8 The Banū Mūsā brothers on a Syrian postage stamp

manuscript that contained only the first seven of the eight books. According to their report[10] the translation was very laborious because the manuscript was full of errors. On several occasions, Ahmad therefore searched for a better copy until he found books I to IV with the commentary by Eutokios in Syria. They now had the *Conica* in two parts of different quality, which they had Hilal *al-Himsi* and Thābit ibn Qurra translate separately. This is how we came to have the second part preserved separately.

The most important translations from Greek were completed around 900 AD; many of these translations also included detailed and expert commentary. We owe the Arabic tradition in particular the preservation of important works by Greek authors:

- Books IV to VII of Diophantus' *Arithmetic* (Qusṭā ibn Lūqā),
- Reconstruction of Euclid's *Data* (Thābit ibn Qurra),
- Completion of Apollonius' *Conica* (Ibn al-Haytham),
- Arabic version of Euclid's *Optics,*
- *On Burning Mirrors* by Diocles.

In addition to the aforementioned mathematical works, philosophical writings by Plato *(Politeia, Nomoi, Timaios)* and Aristotle *(Analytics, Physics)* were also translated.

Author	Works	Translator	Time
Euclid	*Elements*	al-Ḥaǧǧāǧ ben Yūsuf, Isḥaq ibn Ḥunayn	Around 800 Around 850
Archimedes	*On the Sphere and Cylinder,* *On the Measurement of a Circle,* *On the Equilibrium of Planes,* *The Method*	Thābit ibn Qurra	Born 836

[10] G. J. Toomer, Apollonius Conics Book V to VII, Preface to the Conics of the Banu Musa, Berlin 1990, pp. 624–628.

Author	Works	Translator	Time
Apollonius	*Conics*	Ahmad ben Mūsā, Ibn al-Haytham Hilal al-Himsi, Thābit ibn Qurra	Around 830
Diophantus	*Arithmetica*	Qusţā ibn Lūqā	Gest. 912
Menelaus	*Sphaerica*	Ishaq ibn Ḥunayn	Geb. 809
Heron	*Metrica*	Qusţā ibn Lūqā	Gest. 912
Ptolemy	*Almagest*	Thābit ibn Qurra	Geb. 836
Pappus	*Collectio*	al-Sijzi	Um 870

After 900 AD, Baghdad became politically insignificant; the end came in 1258 when the Mongols, under their leader *Hulagus*, conquered Baghdad. From 1405 Baghdad came under Turkish rule.

Epilogue The concluding remarks of J.L. Berggren[11] (1984) are still relevant today:

> The history of Greek mathematics is alive and vibrant. It draws on the cultures of the East - in particular India and the Islamic world - and demands fresh blood, the old texts are read in a new way by a new generation of scholars; the old puzzles continue to fascinate. The challenge of understanding the development of this system of thought has become a case study for mathematics and continues to inspire scholars to their best efforts, so that the history of Greek mathematics remains an important and growing part of the history of science.

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